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Quantification stochastique d'Anderson et calcul paracontrôlé

EDP stochastiques en environnement singulier

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Résumé

Les problèmes d'évolution dirigés par un signal d'entrée sont omniprésents en mathématiques car ils présentent un cadre très général dans lequel on peut modéliser des dynamiques. Dans un cadre de dimension finie, pour l'évolution au fil du temps d'un système X gouverné par des fonctions non linéaires f et g et dirigé par un signal d'entrée ξ, on considère des équations de la forme

$$\frac{dX_t}{dt} = f(X_t, t) + g(X_t, t)\xi_t.$$

Supposons que la donnée ξ soit en fait un bruit, par exemple pour modéliser l'interaction entre des particules dans un milieu désordonné, alors ξ représente un terme de forçage stochastique. En tant que tel, ξ manque de régularité et l'équation ci-dessus sort du cadre de la théorie classique des équations différentielles, c'est le cas du bruit blanc $\xi = \frac{dB}{dt}$ où B est un mouvement brownien. Ce dernier étant nulle part différentiable, cela nécessite une toute autre théorie connue sous le nom de calcul stochastique et de théorie d'Itô. Les équations aux dérivées partielles stochastiques (EDPS) visent à donner un pendant en dimension infinie à de tels modèles, où le système u vit maintenant dans un espace de Banach, où cette fois les termes non linéaires F, G peuvent également impliquer des opérateurs agissant sur le modèle

$$\partial_t u = F[u] + G[u]\xi.$$

Les termes de bruit tels que ξ apparaissant dans les équations sont toujours très irréguliers, étant généralement un bruit blanc, ce qui fait de la résolution de telles équations une véritable difficulté. De plus, en raison de l'irrégularité des fonctions en jeu, des termes mal posés peuvent apparaître dans les équations, par exemple en prenant des fonctionnelles non linéaires d'une distribution. C'est le cas des équations bien connues de Kardar–Parisi–Zhang (KPZ), Φ^4_3 ou du modèle d'Anderson parabolique (PAM) que nous appelons EDPS singulières. Pour ces équations pathologiques, des progrès majeurs ont été observés au cours des 20 dernières années, fournissant de nouveaux et puissants outils pour leur étude. Au prix d'une procédure de *renormalisation*, cela a permis de traiter ces questions inaccessibles auparavant. On ne donne dans ce travail que quelques exemples des nombreuses questions pouvant être traitées avec ces outils à portée de main, le but étant de mettre en avant la philosophie générale derrière le traitement des EDPS singulières via le calcul paracontrôlé.

EDPS singulières : un peu de contexte

Dans un cadre discret, un bruit ξ est défini comme une fonction aléatoire dont la valeur en chaque point est une variable aléatoire. Pour simplifier, supposons qu'un tel bruit ξ soit centré et gaussien, alors il est entièrement déterminé par son noyau de corrélation

$$(x,y) \mapsto \mathbb{E}\Big[\xi(x)\xi(y)\Big].$$

Dans le cas continu, cela peut ne même pas être une fonction, simplement une distribution. C'est le cas du bruit blanc, pour lequel la fonction de covariance est donnée par la masse de Dirac. Autrement dit, un bruit blanc sur $\mathbb{R}^d$ est un processus gaussien centré avec

$$\mathbb{E}\Big[\xi(x)\xi(y)\Big] = \delta_0(x-y),$$

en d'autres termes, ξ est indexé par des fonctions régulières et son noyau de corrélation est donné par le produit scalaire L^2

$$\mathbb{E}\Big[\langle \xi, \varphi\rangle\langle \xi, \psi\rangle\Big] = \int_{\mathbb{R}^d} \varphi(x)\psi(x)\,dx.$$

Par nature, ξ est alors un objet aléatoire avec une régularité locale très limitée, on peut par exemple prouver que $\xi \in \mathcal{C}^{-\frac{d}{2}-\kappa}$ pour tout $\kappa > 0$, illustrant le fait général que *plus il y a de corrélation, plus il y a de régularité*. Un tel bruit (qu'il soit seulement spatial ou espace-temps) apparaît naturellement dans des modèles classiques d'EDPS tels que l'équation KPZ

$$\partial_t u - \partial_x^2 u = \Big(\partial_x u\Big)^2 + \xi,$$

le modèle Φ^4_d

$$\partial_t u - \Delta u = -u - u^3 + \xi,$$

ou l'équation PAM

$$\partial_t u - \Delta u = u\xi.$$

Outre le bruit, ces modèles partagent tous une propriété similaire : ils impliquent une opération (éventuellement non linéaire) qui est mal définie. En effet, en raison de la faible régularité de ξ, on ne peut pas attendre de la solution u qu'elle ait beaucoup de régularité, empêchant les opérations de produit ci-dessus de faire sens. Par exemple, $\partial_x u$ sera une distribution dans le cas de KPZ, de sorte que $\Big(\partial_x u\Big)^2$ n'a pas de sens, de même pour u^3 dans le modèle Φ^4_d lorsque $d \geq 2$, et dans le cas de PAM, on ne s'attend pas à ce que u ait suffisamment de régularité pour pouvoir donner un sens au produit $u\xi$. C'est la partie *singulière* des *EDPS singulières*. La conséquence majeure de cela est qu'il n'y a aucun espoir d'appliquer les outils analytiques habituels pour résoudre ces équations, le sens même à donner à une "solution" d'une telle équation singulière est assez peu clair.

Le manque d'un cadre analytique robuste dans lequel on pourrait donner un sens à ces équations a été résolu en faisant l'observation suivante : les espaces fonctionnels habituels ne conviennent pas à l'analyse des EDPS singulières, elles doivent être étudiées dans des espaces aléatoires appropriés construits à partir du bruit lui-même. C'est l'idée derrière la théorie révolutionnaire des *Structures de Régularité*, introduite par Hairer dans [Hai14], où la régularité locale est mesurée en termes d'objets de référence construits à partir du bruit. L'approche parallèle du *Calcul Paracontrôlé* proposée par Gubinelli, Imkeller et Perkowski dans [GIP15] repose sur des outils d'analyse harmonique pour fournir une description globale des fonctions, mettant l'accent sur leur comportement à haute fréquence où leur régularité est mesurée. Cela a été adapté par Bailleul, Bernicot et Frey dans [BB16], [BBF17] et [BB19], ainsi que par Mouzard dans [Mou22] dans le cadre d'une variété générale, en construisant un calcul paracontrôlé d'ordre supérieur basé sur le semi-groupe de la chaleur. Dans toutes ces techniques, le point central est de donner une description des solutions potentielles u en termes de données de bruit de référence, de sorte que les problèmes singuliers ne mettent en jeu que les objets de référence. La construction de ces objets s'appelle la *procédure de renormalisation* et repose uniquement sur les propriétés probabilistes du bruit. En somme, cela consiste à soustraire une constante infinie dans l'équation pour contrebalancer les termes singuliers.

Prenons par exemple l'équation KPZ introduite dans [KPZ86]

$$\partial_t u - \partial_x^2 u = \left(\partial_x u\right)^2 + \xi$$

sur $\mathbb{R} \times \mathbb{T}$, où ξ est un bruit blanc espace-temps et $\mathbb{T}$ le tore unidimensionnel. Le changement de variable formel $u = \log(v)$, appelé transformée de Hopf-Cole, donne l'équation de la chaleur stochastique suivante sur v

$$\partial_t v - \partial_x^2 v = v\xi.$$

La variable spatiale étant unidimensionnelle, cette équation peut être interprétée comme une équation d'Itô

$$dv_t = \partial_x^2 v_t \, dt + v_t \, dW_t$$

où W est un mouvement brownien cylindrique sur $L^2(\mathbb{T})$. Cette dernière a une unique solution mild grâce aux outils habituels d'EDPS et il semble naturel que $u = \log(v)$ soit alors une solution de l'équation KPZ. Cependant, c'est avec la formule d'Itô qu'il faut déduire l'équation satisfaite par u et ce faisant, on obtient un terme de correction infini, de sorte que u vérifie plutôt

$$\partial_t u - \partial_x^2 u = \left(\partial_x u\right)^2 - \infty + \xi$$

au moins formellement (ce calcul formel peut être rendu précis en remplaçant W par une version régularisée W_ε et en étudiant la limite $\varepsilon \to 0$).
Tant la théorie des structures de régularité que le calcul paracontrôlé ont été fructueux depuis leurs débuts. On notera les travaux de Bailleul et Hoshino [BH21a] et [BH21b] où ils proposent un dictionnaire pour faire la traduction entre les structures de régularité et le calcul paracontrôlé. Notons également le travail de Chauleur et Mouzard [CM23] où ils étudient l'équation de Schrödinger non linéaire logarithmique, étant capables de manipuler une non-linéarité non localement lipschitzienne.

Le cas quasilinéaire des EDPS singulières a fait l'objet de nombreux développements grâce à ces outils, voir par exemple les travaux [BDH16; FG19; BHK23; OW19; OSSW23; LOT23]. Dans le Chapitre V, on étudie une version non locale de l'équation quasilinéaire PAM sur le tore bidimensionnel. En utilisant les outils du calcul paracontrôlé et leur interaction avec les opérateurs pseudo-différentiels tels qu'étudiés dans [BDFT23b], on suit la même démarche que dans [BDH16] pour prouver le caractère bien posé localement.

L'Hamiltonien d'Anderson, un opérateur singulier

Dans le cas d'un bruit blanc multiplicatif spatial, comme c'est le cas pour l'équation PAM linéaire, on peut même considérer le produit singulier $u\xi$ comme faisant partie de l'opérateur qui régit la dynamique. Cela donne formellement l'opérateur

$$\mathcal{H} = -\Delta + \xi$$

appelé l'*Hamiltonien d'Anderson*. En raison de l'irrégularité de ξ, $\mathcal{H}$ ne rentre pas dans le cadre de la théorie des perturbations habituelle en dimension $d \geq 2$, mais il s'avère que les outils mentionnés précédemment sont parfaitement adaptés à l'étude d'un tel opérateur singulier. Il est défini pour la première fois dans [And58] comme un moyen de modéliser un phénomène de concentration de masse pour les systèmes quantiques soumis à une force aléatoire, c'est le phénomène de *localisation d'Anderson*. Dans un cadre discret, on renvoie au livre de König [Kön16] pour une étude complète. En ce qui concerne la dimension 1, le bruit est suffisamment régulier pour que l'opérateur fasse directement sens, en suivant par exemple la construction de Fukushima et Nakao [FN77]. Cela ne signifie pas pour autant qu'il n'y a aucune difficulté et les travaux récents de Dumaz et Labbé [DL20], [DL22] et [DL24] fournissent une compréhension très complète des propriétés de l'opérateur dans un cadre unidimensionnel. Dans [DL24], ils proposent une plage complète de régimes pour les propriétés spectrales de $\mathcal{H}$ centré autour d'un niveau d'énergie E, en fonction de la manière dont E se compare à la taille du segment sur lequel l'opérateur est construit. Dans [DL22], ils établissent les propriétés spectrales de $\mathcal{H}$ défini sur toute la droite réelle, récupérant la localisation d'Anderson : la mesure spectrale de l'opérateur est presque sûrement pure-point et les fonctions propres sont exponentiellement localisées.

Les choses se compliquent en dimension 2 car le produit avec ξ est alors singulier. La première construction revient à Allez et Chouk dans [AC15] où ils parviennent à construire l'opérateur sur le tore $\mathbb{T}^2$. L'idée était d'introduire un espace de fonctions paracontrôlées comme domaine de $\mathcal{H}$ et de le définir par une procédure de renormalisation. Ils parviennent à définir $\mathcal{H}$ comme un opérateur auto-adjoint non borné sur $L^2(\mathbb{T}^2)$ avec un spectre discret, ainsi qu'à obtenir des bornes sur les valeurs propres dans la limite grand volume. Labbé a obtenu un résultat similaire dans [Lab19] en utilisant les outils des structures de régularité, définissant l'opérateur dans des domaines bornés en dimension $d \leq 3$. Dans [GUZ20], Gubinelli, Ugurcan et Zachhuber ont construit l'opérateur sur $\mathbb{T}^2$ et $\mathbb{T}^3$ en utilisant une combinaison du calcul paracontrôlé et de la transformée exponentielle pour gérer la construction tridimensionnelle. En utilisant les outils du calcul paracontrôlé d'ordre supérieur, Mouzard a construit dans [Mou22] l'opérateur dans le cadre général d'une surface riemannienne fermée (compacte, connexe, sans bord) $\mathcal{M}$, prouvant qu'il peut être défini comme un opérateur auto-adjoint sur $L^2(\mathcal{M})$ avec un domaine aléatoire constitué de fonctions paracontrôlées (comme dans [AC15], [Lab19] ou [GUZ20]). Il établit que l'opérateur a un spectre discret et que ses valeurs propres satisfont une loi de Weyl. Une construction plus simple en dimension 2 et 3 via sa forme quadratique peut également être trouvée dans [MO23]. Enfin, dans [BDM23], Bailleul, Dang et Mouzard fournissent une construction de $\mathcal{H}$, sur une surface générale, basée sur la construction de sa résolvante. Ils parviennent à retrouver les propriétés des travaux mentionnés précédemment ainsi qu'une compréhension fine du semigroupe de la chaleur correspondant $e^{-t\mathcal{H}}$ et de son noyau. Ils obtiennent des propriétés très proches de celles que l'on pourrait attendre pour une perturbation habituelle de l'opérateur de Laplace en utilisant l'application Γ qui décrit le domaine (aléatoire) de $\mathcal{H}$.

Ayant l'opérateur et ses propriétés à disposition, on peut essayer de transférer les résultats habituels de $-\Delta$ à $\mathcal{H}$. Par exemple, les équations de Schrödinger et des ondes sur des domaines périodiques ont été étudiées dans [GUZ20]. Zachhuber dans [Zac20] a obtenu des estimées de Strichartz pour l'équation de Schrödinger sur $\mathbb{T}^2$ en présence d'un bruit blanc espace, avec Mouzard dans [MZ22] ils améliorent ce résultat au cas des surfaces riemanniennes, obtenant des estimations de Strichartz pour $\mathcal{H}$ dans le contexte de l'équation de Schrödinger et de l'équation des ondes. Debussche et Weber dans [DW18], Debussche et Martin dans [DM19] puis Tzvetkov et Visciglia dans [TV23b] et [TV23a] étudient l'équation de Schrödinger avec un bruit blanc spatial multiplicatif en utilisant la *transformée exponentielle* des travaux de Hairer et Labbé, prouvant le caractère bien posé globalement, pour des données initiales bien préparées. Avec Debussche et Liu dans [DLTV23], ils étendent ces résultats au cas du plan entier. Dans le cas des équations elliptiques, Zhang et Duan ont étudié dans [ZD23] la régularité des solutions de l'équation de Schrödinger stationnaire non linéaire associée à $\mathcal{H}$.

Concernant le cas du volume infini en dimension 2, le manque de compacité rend les choses encore plus difficiles et il faut travailler dans des espaces à poids pour contrôler le comportement à l'infini. Dans [Uek], Ueki définit l'opérateur sur $\mathbb{R}^2$ en utilisant des techniques similaires à celles de [Mou22]. Enfin, dans le travail [HL24], Hsu et Labbé parviennent à définir l'opérateur dans l'espace complet $\mathbb{R}^2$ et $\mathbb{R}^3$ en basant leur analyse sur la théorie de l'équation PAM posée sur $\mathbb{R}^2$ et $\mathbb{R}^3$.

Alors que nous rappelons quelques bases sur la construction de $\mathcal{H}$ sur une surface fermée dans le Chapitre I, le but principal de ce chapitre est de fournir un résultat supplémentaire pour convaincre que $\mathcal{H}$ peut être vu comme une perturbation du laplacien. En ce sens, on donne une comparaison quantitative entre les fonctions de Green de $\mathcal{H}$ et de $-\Delta+1$, sur laquelle on basera notre analyse dans les Chapitres III et IV.

Dans le Chapitre II, on utilise certaines des propriétés de $\mathcal{H}$ pour étudier des équations elliptiques associées dans un cadre compact bidimensionnel. Contrairement à [ZD23], on ne s'attarde pas sur la régularité des solutions, mais plutôt sur l'adaptation des outils de l'analyse non linéaire au cadre des opérateurs singuliers aléatoires. On adapte les méthodes dites de *Mountain Pass* pour assurer l'existence de multiples solutions à l'équation de Schrödinger stationnaire en présence d'un bruit blanc spatial, ainsi que certains outils d'analyse convexe pour gérer des non-linéarités de type Hartree dans un contexte similaire.

Mesures de Gibbs et théorie quantique des champs

La procédure de renormalisation sur laquelle repose la discussion ci-dessus n'est pas inhérente au cadre du calcul paracontrôlé ou des structures de régularité, c'est un outil naturel lors de l'étude des questions des mesures invariantes et de théorie quantique des champs. On y voit apparaître naturellement des EDPS singulières, en cherchant à fournir une description fine des systèmes de particules fondamentales interagissant entre elles.

Dans un cadre discret, soit Λ un ensemble fini de sites, une configuration (ou champ) est alors une fonction réelle $u : \Lambda \to \mathbb{R}$. Supposons que le système est régi par une énergie donnée par

$$\mathcal{E}(u) = \frac{1}{2}\sum_{x\sim y}|u(x)-u(y)|^2 + \frac{1}{2}\sum_x |u(x)|^2 + \sum_x V(u(x))$$

avec une énergie potentielle V, où on écrit $x \sim y$ lorsque $x, y \in \Lambda$ sont suffisamment "proches" pour interagir entre eux. Le but est alors de comprendre la mesure de Gibbs correspondante

$$d\rho(u) \propto e^{-\mathcal{E}(u)}\prod_x du(x).$$

En considérant la partie quadratique de l'énergie avec $\Pi_x du(x)$, on obtient une nouvelle mesure de probabilité de référence μ appelée *Champ Libre Gaussien* (GFF) (ou plus précisément, une variable aléatoire avec loi μ est appelée un GFF), de sorte que finalement

$$d\rho(u) \propto e^{-\sum\limits_x V(u(x))} d\mu(u).$$

Cela ne pose bien sûr aucun problème dans le cas où Λ est fini, cependant la question se pose de donner un sens à une telle mesure dans le cadre continu. En effet, si le terme d'énergie potentielle consiste en une fonction non linéaire d'une distribution u, on est face au même problème définition que précédemment. Même s'il n'y a pas de mesure de Lebesgue dans les espaces de dimension infinie, la partie quadratique peut elle par contre nous aider. On peut toujours définir μ comme une mesure gaussienne. L'équivalent continu de la discussion ci-dessus revient à considérer l'énergie cinétique

$$\frac{1}{2}\langle u, (-\Delta+1)u\rangle_{L^2}$$

et donc à la définition d'un GFF dans le cadre continu comme un processus aléatoire gaussien centré, indexé par des fonctions régulières, et ayant pour opérateur de covariance $(-\Delta+1)^{-1}$ (ou de façon équivalente, ayant la fonction de Green de $-\Delta+1$ comme noyau de covariance). Dans un domaine compact en dimension 2, par exemple, la mesure $\mu^{-\Delta+1}$ du GFF peut également être définie comme la loi de la série aléatoire

$$u^\omega(x) = \sum_{n\geq 0}\frac{\gamma_n(\omega)}{\sqrt{1+\lambda_n^\Delta}}\varphi_n^\Delta(x)$$

où $(\lambda_n^\Delta, \varphi_n^\Delta)$ sont les valeurs propres et fonctions propres de Δ et γ_n sont des variables aléatoires gaussiennes standard i.i.d.. Cette série converge dans $H^{-\kappa}$ pour tout $\kappa > 0$, mais pas dans L^2, prouvant que les éléments génériques dans le support de $\mu^{-\Delta+1}$ sont des distributions, pas des fonctions. Cela soulève la question de donner une définition appropriée d'une densité par rapport à la loi du GFF dès lors que le potentiel V est non linéaire. Par exemple, lorsque $V = \frac{1}{4}|\cdot|^4$ et que l'espace ambiant est le tore $\mathbb{T}^d$, on

retrouve la mesure Φ^4_d
$$d\rho(u) \propto e^{-\frac{1}{4}\int_{\mathbb{T}^d}|u(x)|^4\,dx} d\mu^{-\Delta+1}(u)$$
qu'on peut voir (au moins formellement) comme la mesure invariante de l'équation Φ^4_d mentionnée précédemment
$$\partial_t u - \Delta u = -u - u^3 + \zeta$$
où ζ est un bruit blanc espace-temps, notons que l'équation est singulière en dimension $d \geq 2$. L'idée de considérer la mesure Φ^4 comme la mesure invariante de la dynamique correspondante remonte à Parisi et Wu dans [PW81] et est le point central de la *Quantification Stochastique*, traduisant le problème de compréhension d'une mesure en l'étude d'une EDPS singulière. Des progrès significatifs ont été réalisés depuis, en particulier depuis le travail de 2003 de Da Prato et Debussche [DD03] où ils obtiennent des solutions fortes (au sens probabiliste) à l'équation de quantification stochastique sur $\mathbb{T}^2$ avec une non-linéarité polynomiale. Pour ne citer que quelques travaux autour de la construction dynamique de la mesure Φ^4, Mourrat et Weber dans [MW17b], Hairer et Mattingly dans [HM17], Albeverio et Kusuoka dans [AK20] ont construit la dynamique et la mesure associée Φ^4_3 sur le tore $\mathbb{T}^3$. Voir aussi la construction de Jagannath et Perkowski dans [JP21]. Pour ce qui est du volume infini, Mourrat et Weber dans [MW17a] ont prouvé le caractère bien posé global du modèle dans le plan, et en 3 dimensions, on se réfère aux travaux de Gubinelli et Hofmanová [GH19], Barashkov et Gubinelli dans [BG20] et [BG23] ou Albeverio et Kusuoka dans [AK22]. Bien que tous ces travaux se déroulent soit dans l'espace euclidien, soit sur le tore, il convient également de noter que des résultats similaires existent dans le cas de variétés riemanniennes. Par exemple, dans le cas d'une surface riemannienne fermée, Burq, Thomann et Tzvetkov dans [BTT18] construisent la mesure $P(\Phi)_2$ et Oh, Robert, Tzvetkov et Wang dans [ORTW20] avec une non-linéarité exponentielle. En ce qui concerne le modèle Φ^4 sur des variétés tridimensionnelles, Bailleul, Dang, Ferdinand et Tô ont construit la mesure Φ^4_3 dans les travaux récents [BDFT23a] et [BDFT23b], et Bailleul dans [Bai23] a également prouvé l'unicité de la mesure invariante.

Dans le Chapitre III, on construit la mesure Anderson-$P(\Phi)_2$ sur une surface riemannienne fermée pour des non-linéarités polynomiales défocalisantes. En d'autres termes, on étudie l'équation Φ_2, avec une non-linéarité polynomiale et perturbée par un bruit blanc spatial indépendant ξ. Cela revient en un sens à étudier l'équation Φ_2 habituelle dans un environnement spatial aléatoire et singulier. La dynamique correspondante est alors dirigée par l'opérateur d'Anderson $\mathcal{H}$, pour laquelle on obtient une mesure de Gibbs invariante décrite par une densité renormalisée par rapport à la mesure GFF d'Anderson. La construction de la mesure repose sur la formule de Boué-Dupuis telle qu'utilisée dans [BG20] et [BG23] pour garantir la convergence de la procédure de renormalisation. Le Chapitre IV s'inscrit comme suite directe de ce travail. En suivant un schéma proche de [MW17b] et [TW18], on tente de construire la mesure précédente directement par la dynamique définie par l'équation. Un telle construction permet notamment d'obtenir les propriétés probabilistiques fortes de la mesure. À noter que dans leur travail très récent [BDZ23], Barashkov, De Vecchi et Zacchuber construisent également une mesure Anderson-Φ^4_2 dans le contexte de l'équation des ondes cubique.

De telles mesures de Gibbs invariantes peuvent également être utilisées pour étudier les équations dispersives correspondantes. C'est l'idée derrière le travail de Lebowitz, Rose et Speer [LRS88] : utiliser la mécanique statistique pour mieux comprendre l'équation. Quelques années plus tard, Bourgain dans [Bou94] sur le cercle et [Bou96] sur $\mathbb{T}^2$ a mis en place ces idées pour étudier l'équation de Schrödinger non-linéaire (respectivement avec une non-linéarité de focalisante et de défocalisante)

$$i\partial_t u + \Delta u = \pm |u|^2 u$$

en partant d'une donnée initiale peu régulière. Le problème que présente l'équation de Schrödinger est que, contrairement à l'équation de la chaleur, le semi-groupe correspondant ne régularise pas. On doit généralement recourir aux inégalités de Strichartz pour obtenir des estimations sur la solution, ce qui réduit fortement l'ensemble des données initiales pour lesquelles le problème est globalement bien posé. L'idée révolutionnaire de Bourgain était, en l'absence de bonnes quantités conservées pour garantir le caractère globale, d'utiliser les mesures de Gibbs invariantes comme moyen de globaliser les solutions. L'idée est alors de formuler un argument de globalisation valable pour les données initiales distribuées selon la mesure de Gibbs pour récupérer une globalisation presque-sûre. Les dynamiques laissant invariantes la mesure Φ_d ont depuis été étudiées dans divers contextes, par exemple pour les équations des ondes dans [GKO18; GKOT22; ORT23; OT20; OTWZ22], les équations de Schrödinger [BL23; DNY19; OT18; Tzv08], et les équations de Ginzburg-Landau complexes [DDF21; RZ; Tre19], pour n'en citer que quelques-unes.

Dans [BTT18] et [OT20], l'accent est mis non pas sur le caractère bien posé de la dynamique, pouvant être une question relativement complexe, mais plutôt sur la question de savoir si on peut assurer l'existence de solutions globales pour des données initiales peu régulières. Après avoir construit une mesure de Gibbs invariante pour l'équation via une procédure de renormalisation, l'idée est d'utiliser un critère de compacité pour établir l'existence d'une solution globale ayant la mesure invariante comme loi à chaque instant. L'argument de compacité prend la forme de l'utilisation successives des théorèmes de Prohorov et Shorokhod après avoir raffiné la mesure de Gibbs en une mesure sur les fonctions espace-temps.

Perspectives

Cette thèse ne présente qu'une fraction des nombreux problèmes intéressants qui peuvent être abordés grâce aux outils du calcul paracontrôlé. Les résultats obtenus ici ouvrent la voie à de nouveaux développements. Par exemple, le Chapitre II amène la question des solitons pour l'équation de Schrödinger-Anderson non-linéaire posée dans l'espace tout entier. Sur la droite réelle, utiliser l'analyse fournie dans [DL22] donne de l'espoir, mais le manque de compacité est un problème majeur.

Une question naturelle suivant les Chapitres III et IV est l'étude des propriétés plus fines de la mesure de Gibbs qui y est construite. Dans le travail en cours sur lequel est basé le Chapitre IV, on montre que le semi-groupe de transition correspondant vérifie la propriété de strong Feller, on est également à même de prouver que la mesure de Gibbs construite précédemment est en fait l'unique mesure invariante pour la dynamique, ainsi que l'ergodicité du système. Un schéma de preuve est donné en Sous-section IV.3.2

Comme la construction du Chapitre III est relativement robuste, une poursuite serait

d'étudier le cas d'une non-linéarité analytique (retrouvant par exemple la théorie de Liouville ou de sine-Gordon). De plus, la méthode utilisée ici semble pouvoir être appliquée comme une sorte de boîte noire valable pour tout opérateur ayant de suffisamment bonnes propriétés.
Dans un autre travail en cours, on étudie le pendant dispersif de l'équation du Chapitre IV. On donne ici l'idée générale derrière le traitement de cette équation. Étant donné un bruit blanc espace ξ on s'intéresse à l'équation de Schrödinger perturbée par ξ

$$i\partial_t u + \Delta u = |u|^2 u + \xi u$$

posée sur le tore $\mathbb{T}^2$, pour des données initiales u_0 peu régulières. Même en l'absence de ξ, l'équation de Schrödinger non linéaire présente des problème à cause de l'absence de régularisation du semi-groupe associé $e^{-it\Delta}$. On ne se préoccupe pas ici du caractère bien posé de l'équation, mais plutôt de savoir si on peut assurer l'existence de solutions globales, même partant d'une donnée initiale peu régulière. L'outil privilégié dans cette situation est la construction d'une mesure de Gibbs invariante pour l'équation formellement donnée par

$$d\rho(u) \propto e^{-\frac{1}{2}\int_{\mathcal{M}} |\nabla u|^2 + \xi u^2 \, dx - \int_{\mathcal{M}} |u|^4 \, dx} \, du.$$

La construction de ρ est analogue à celle effectuée au Chapitre I, à ceci près qu'on doit gérer des fonctions à valeurs complexes. ρ est alors définie comme limite en variation totale d'une suite de mesures ρ_N à densité par rapport à la loi du champ libre gaussien $\mu^{\mathcal{H}}$ de $\mathcal{H}$. L'idée est alors de raffiner la suite de mesures $(\rho_N)_N$ en une suite de mesures $(\nu_N)_N$ portant sur les fonctions espace temps

$$\nu_N = \rho_N \circ \Phi_N^{-1}$$

où Φ_N est le flot de l'équation tronquée pour laquelle ρ_N est invariante. On montre alors que la suite $(\nu_N)_N$ est tendue et, en utilisant les théorèmes de Prokhorov et Skorokhod, on construit une suite de fonctions $(u_N)_N$ distribuées suivant $(\nu_N)_N$ qui converge vers une solution globale du problème initial.

Contenu de la thèse

Chapitre I : Opérateur d'Anderson et calcul paracontrôlé

On rappelle dans ce chapitre le contexte analytique dans lequel se situera la suite de ce travail. Le point de départ du calcul paracontrôlé se situe dans la décomposition de Paley-Littlewood, pour une distribution $u \in \mathcal{D}'(\mathbb{T}^d)$

$$u = \sum_{j \geq -1} \Delta_j u$$

où les blocs de Paley $\Delta_j u$ sont localisés autour de fréquences de taille $\approx 2^j$. On peut alors écrire formellement le produit de deux distributions u et v sous la forme

$$\begin{aligned} uv &= \sum_{i<j-1} \Delta_i u \Delta_j v + \sum_{|i-j|\leq 1} \Delta_i u \Delta_j v + \sum_{j<i-1} \Delta_i u \Delta_j v \\ &= \mathsf{P}_u v + \mathsf{\Pi}(u,v) + \mathsf{P}_v u \end{aligned}$$

où on désigne par $(\mathsf{P}, \mathsf{\Pi})$ la paire paraproduit-résonance. Tandis que les termes de parapoduit $\mathsf{P}_u v$ et $\mathsf{P}_v u$ sont toujours bien définis, le terme résonant $\mathsf{\Pi}(u,v)$ capture la singularité du produit uv. Ainsi, la difficulté première des EDPS singulières consiste à fournir une interprétation analytique de ce terme à-priori mal défini sans hypothèse de régularité forte sur u ou v. Prenant l'exemple de l'équation de la chaleur forcée par un bruit blanc spatial ξ

$$\partial_t u - \Delta u = f(u) + u\xi$$

posée sur $\mathbb{R}_+ \times \mathbb{T}^2$, le produit $u\xi$ est mal défini à cause de la forte irrégularité de ξ. À l'aide des outils du calcul paracontrôlé, on est toutefois à même de donner un sens à l'équation en prescrivant une structure aux fonctions u que l'on cherche

$$u = \mathsf{P}_u X + u^\sharp$$

où X est une donnée ne dépendant que du bruit ξ et $u^\sharp$ est un terme de reste ayant une meilleure régularité. Après une procédure de renormalisation purement probabiliste, on est alors ramené à un cadre dans lequel on peut espérer mener une étude à l'aide des outils analytiques usuels.
Le modèle d'équations qui retiendra notre attention tout au long de ce travail est celui de dynamiques dirigées par l'opérateur aléatoire

$$\mathcal{H} = -\Delta + \xi$$

appelé opérateur d'Anderson. On lui donne un sens en tant qu'opérateur non-borné sur L^2, dont le domaine est aléatoire, donné par une décomposition similaire à celle évoquée plus haut, avec une bonne théorie spectrale. $\mathcal{H}$ peut être pensé comme une perturbation du laplacien, mais les outils classiques de perturbation ne sont pas accessibles à cause de l'irrégularité de ξ, on parvient cependant à montrer de nouveaux résultats en ce sens. Notamment en assurant que les fonctions de Green de $\mathcal{H}$ et de $-\Delta+1$ ont exactement la même singularité, ou en fournissant une comparaison quantifiée entre les symboles basés sur chacun de ces opérateurs.

Chapitre II : EDP elliptiques singulières

• [BER22], I. Bailleul, H. Eulry et T. Robert, *Variational methods for some singular stochastic elliptic PDEs*, à paraître aux *Annales de la Faculté des Sciences de Toulouse.*

Ayant effectué la construction de l'opérateur d'Anderson, on s'intéresse à des équations stationnaires dirigées par $\mathcal{H}$ posées sur une surface compacte $\mathcal{M}$. À l'aide d'outils d'analyse non-linéaire, on étudie une équation elliptique singulière qu'on ne peut résoudre avec une méthode de point fixe classique, contraction ou Schauder. Plus précisé-

ment, étant donnée une non-linéarité f et un potentiel $a \in L^p(\mathcal{M})$, $p > 1$, on considère l'équation suivante

$$(-\Delta + a)u = f(\cdot, u) + \xi u.$$

Sous des hypothèses sur a et f, les théorèmes de points fixes habituels permettent d'assurer l'existence de solutions, mais cela nécessite d'une façon ou d'une autre une hypothèse de petitesse sur a et d'intégrabilité sur f. Dans le cas général, en caractérisant les solutions comme points critiques d'une certaine fonctionnelle d'énergie

$$\Phi(u) := \frac{1}{2}\|u\|_{\mathcal{D}^1}^2 + \int_{\mathcal{M}} \left(\frac{1}{2}a(x)u(x)^2 - F\Big(x, u(x)\Big)\right) dx$$

on utilise alors des méthodes de compacité de type Mountain-Pass. Ceci requiert de s'intéresser aux perturbations de l'opérateur $\mathcal{H}$ par des potentiels intégrables, ce qui donne lieu à la notion de classe de Kato pour l'opérateur et assure essentiellement que $\mathcal{H} + a$ conserve la même théorie spectrale que $\mathcal{H}$.
On montre alors que le problème elliptique

$$(\mathcal{H} + a)u = f(\cdot, u)$$

admet au moins une solution dès lors que a et f satisfont des hypothèses raisonnables, incluant par exemple les monômes $f(x, u) = u|u|^\ell$. Sous des hypothèses supplémentaires de symétrie, vérifiées par exemple si ℓ est pair, on montre même que l'équation admet une infinité de solutions, ce qui contraste fortement avec les résultats usuels d'existence et unicité.
Ayant à disposition la théorie spectrale de $\mathcal{H} + a$, on peut également traiter le cas d'équations qui ne peuvent pas se formuler comme une recherche de points critiques de fonctionnelles d'énergie. C'est le cas de la version singulière de l'équation de Choquard-Pekar sur le tore de dimension 2

$$(-\Delta + a)u = \Big(w \star f(u)\Big)g(u) + \xi u,$$

où $\star$ désigne l'opération de convolution. On utilise dans ce cas la machinerie des fonctionnelles auto-duales de Ghoussoub pour obtenir l'existence d'une solution à cette équation, sous la forme d'un point atteignant le minimum d'une fonctionnelle auto-duale fortement coercive sous certaines hypothèses sur les coefficients a, w, f, g de l'équation.
Un des principaux intérêts de l'étude de ce type d'équations et d'obtenir l'existence de solutions stationnaires pour les équations de Schrödinger associées. Précisément, prenant l'exemple d'une non-linéarité polynomiale, les résultats précédents assurent l'existence d'ondes solitaires (ou solitons) pour l'équation de Schrödinger posée sur $\mathbb{R} \times \mathcal{M}$

$$i\partial_t v + \mathcal{H}v = |v|^{p-1}v$$

sous la forme $v(t, x) = e^{i\lambda t}u(x)$ où u vérifie

$$\mathcal{H}u + \lambda u = |u|^{p-1}u.$$

Le même raisonnement s'applique dans le cas d'une non-linéarité non-locale comme plus haut

$$i\partial_t v + \mathcal{H}v = \Big(w \star |v|^p\Big)|v|^{q-2}v$$

où l'on obtient v sous la forme $v(t,x) = e^{i\lambda t}u(x)$ avec u satisfaisant l'équation elliptique associée

$$\mathcal{H}u + \lambda u = \Big(w \star |u|^p\Big)|u|^{q-2}u.$$

Une telle solution u conserve alors un profil constant en temps, modulé par des oscillations indépendantes de la variable d'espace.

Chapitre III : Mesures de Gibbs invariantes en environnement singulier

- [EMR24], H. Eulry, A. Mouzard et T. Robert, *Anderson stochastic quantization equation*, preprint ArXiv:2401.12742.

On s'intéresse dans ce chapitre à l'équation de quantification sur une surface compacte $\mathcal{M}$, forcée à la fois par un bruit blanc espace multiplicatif ξ et un bruit blanc indépendant espace-temps additif ζ

$$\partial_t u - \Delta u + \xi u - F'(u) = \sqrt{2}\zeta$$

où F est une non-linéarité polynomiale défocalisante. On a alors à gérer deux couches d'aléa et de singularité. D'une part le modèle $P(\Phi)_2$ en l'absence de ξ est déjà en lui-même singulier à cause de la mauvaise régularité de ζ et nécessite une étude fine via une procédure de renormalisation de l'équation. D'autre part l'opérateur d'Anderson $\mathcal{H} = -\Delta + \xi$ est lui aussi singulier. On s'appuie dès lors de façon cruciale sur la théorie de résolution pour $\mathcal{H}$. On en appelle alors aux résultats sur le semi-groupe de la chaleur de l'opérateur d'Anderson, sa théorie spectrale et sa fonction de Green, évoqués au Chapitre I. Toute la machinerie repose sur la décomposition de ζ dans une base bien choisie de $L^2(\mathcal{M})$

$$\zeta(ds) = \sum_{n\geq 0} dB_n(s)\varphi_n^{\mathcal{H}}$$

où $(B_n)_{n\geq 0}$ est une famille i.i.d. de browniens réels. On voit alors sur cette décomposition la double couche d'aléa s'affronter, d'une part l'aléa de $\mathcal{H}$ au travers de ses fonctions propres $\varphi_n^{\mathcal{H}}$, d'autre part l'aléa de ζ au travers des coefficients B_n.
L'étude de cette dynamique a pour intérêt principal de proposer un modèle parabolique pour la construction d'une mesure de Gibbs associée

$$d\rho(u) \propto e^{-\frac{1}{2}\int_{\mathcal{M}} |\nabla u|^2 + \xi u^2\,dx + \int_{\mathcal{M}} F(u)\,dx}\,du$$

En considérant séparément la partie quadratique, ρ est alors définie comme une mesure à densité par rapport au champ libre gaussien $\mu^{\mathcal{H}}$ associé à $\mathcal{H}$. Ce dernier étant à support dans des espaces de distributions, il est nécessaire de passer par une procédure de renormalisation pour faire sens du terme d'énergie potentielle $\int_{\mathcal{M}} F(u)\,dx$. La construction de ρ repose alors de façon cruciale sur la formule de Boué-Dupuis et un passage à la limite dans ses versions renormalisées ρ_N.
La mesure de Gibbs, candidate pour une mesure invariante à l'équation, étant construite à densité par rapport à $\mu^{\mathcal{H}}$, on cherche alors à étudier finement les propriétés de

l'équation avec une donnée initiale dans le support de $\mu^{\mathcal{H}}$. L'étude repose sur une décomposition astucieuse de u en suivant l'idée de Da Prato et Debussche. En notant ⫯ la convolution stochastique, i.e. la solution stationnaire de

$$\partial_t \text{⫯} + \mathcal{H}\text{⫯} = \sqrt{2}\zeta,$$

alors $v = u - \text{⫯}$ satisfait l'équation non-linéaire

$$\partial_t v + \mathcal{H}v + F'(v + \text{⫯}) = 0$$

dont on attend que les solutions aient une meilleure régularité que celle de u. L'étude de ⫯ nécessite un traitement particulier pour en définir les puissances renormalisées, ⫯ étant à-priori une distribution, on ne peut pas de façon directe en définir une fonctionnelle non-linéaire. Ce traitement est similaire à celui qui doit être fait pour le terme d'énergie potentielle. On obtient alors un résultat d'existence de solutions en temps court, ainsi que la convergence du modèle renormalisé vers l'équation limite.
On prouve enfin que la mesure ρ est invariante par la dynamique et que les solutions existent globalement pour des données initiales distribuées suivant cette mesure. L'argument clé de la globalisation, en l'absence de loi de conservation, étant d'utiliser la mesure invariante elle-même comme invariant, on obtient alors une borne en espérance

$$\int_{\mathcal{C}^{-\varepsilon}\times\Omega} \sup_{0\leq t\leq T\wedge T^*} \|\Phi(u_0,\omega)(t)\|_{\mathcal{C}^{-\varepsilon}}\, d\rho(u_0)\, d\mathbb{P}(\omega) \leq C_T$$

sur le flot de l'équation partant de u_0, ce qui suffit à assurer le caractère global presque sûrement pour des conditions initiales distribuées suivant ρ.

Chapitre IV : Mesure Φ^4_2 en environnement singulier

- A. Debussche, H. Eulry et A. Mouzard, *Ergodicity for the Anderson Φ^4_2 measure*, en preparation.

Ce chapitre est la suite directe du précédent, on y considère l'équation Φ^4_2 posée sur $\mathbb{T}^2$ et perturbée par un bruit blanc espace

$$\partial_t u + \mathcal{H}u + u^3 = \sqrt{2}\zeta$$

partant d'une donnée initiale u_0. La théorie locale pour cette équation ayant été traitée précédemment, on s'intéresse ici putôt au caractère global des solutions ainsi qu'aux propriétes probabilistes de la solution obtenue. En particulier, on cherche à construire la mesure précédente de façon dynamique. En écrivant une fois de plus $u = \text{⫯} + v$ comme précédemment, on transforme l'équation en

$$\partial_t v + \mathcal{H}v + (v + \text{⫯})^3 = 0.$$

Cette fois cependant, on fait commencer ⫯ en 0 au temps initial de sorte que u et v aient la même condition initiale. Pour obtenir des bornes sur v et espérer pouvoir globaliser, on cherche des bornes d'énergie sur v. Cependant, l'argument usuel de tester l'équation contre une puissance de v s'avère ici complexe, à cause du caractère singulier de $\mathcal{H}$. En utilisant l'application Γ de la construction de $\mathcal{H}$ on est à même de lever cette difficulté. Comme v satisfait une équation dirigée par $\mathcal{H}$, v admet une structure paracontrôlée et on peut écrire $v = v \varolessthan X + w$ où $\Delta X = \xi$. L'application $v \mapsto w$ est alors inversible (modulo un bon choix du paraproduit $\varolessthan$) et on ramène la question de l'existence globale à l'étude de l'équation posée sur w

$$\partial_t w - \Delta w + w^3 + R(w) + Q(w) = 0,$$

où $R = \Gamma^{-1}\mathcal{H}\Gamma + \Delta$ est la correction pour passer de $\mathcal{H}$ à $-\Delta$ et perd 1^+ dérivées. Cette perte de dérivée était attendue et provient de l'irrégularité de l'opérateur $\mathcal{H}$. Q quant à lui regroupe les termes provenant de la non-linéarité cubique, faisant intervenir la donnée du bruit ζ au travers de Ψ et de ses puissances renormalisées, c'est à dire

$$Q(w) + w^3 = \Gamma^{-1}\left(\left((\Gamma w) \varolessthan X + w + \Psi\right)^3\right).$$

Le terme $Q(w)$ regroupe alors tous les termes d'homogénéité inférieure ou faisant intervenir l'opérateur régularisant $\cdot \varolessthan X$, ce qui permet de considérer Q comme un terme perturbatif. Là où le signe devant la non-linéarité ne joue pas de rôle dans la théorie locale, il est ici crucial et permet de contrôler les termes provenant de R et Q. On obtient alors une borne à-priori sur les solutions w de l'équation ne faisant pas intervenir la condition initiale et ne dépendant du bruit que sur l'intervalle de temps courant. Par un argument de *coming down from infinity*, on est alors en mesure de prouver que les solutions u existent globalement, pour toute donnée initiale u_0 dans $\mathcal{C}^{-\varepsilon}$ ainsi qu'une borne sur les moments de la solution

$$\sup_{u_0 \in \mathcal{C}^{\varepsilon}} \sup_{t \geq 0} \left(1 \wedge t^{\frac{3p-2}{2}}\right) \mathbb{E}\left[\|u(t;u_0)\|_{\mathcal{C}^{-\varepsilon}}^{3p-2}\right] < +\infty.$$

u étant solution d'une équation dirigée par un bruit blanc espace-temps, on peut en déduire des propriétes sur le processus $t \mapsto u(t;\cdot)$ en tant que processus à valeurs dans $\mathcal{C}^{-\varepsilon}$. Notant $(P_t)_{t \geq 0}$ son semigroupe de transition défini par

$$P_t\Phi(u_0) = \mathbb{E}\left[\Phi\Big(u(t;u_0)\Big)\right],$$

alors $(P_t)_{t\geq 0}$ est un semigroupe de Markov pour la filtration usuelle associée à ζ. On montre également qu'il possède la propriété de Feller en tant que semigroupe agissant sur $\mathcal{C}^{-\varepsilon}$. Notant $(P_t^*)_{t\geq 0}$ son semigroupe dual, on montre enfin à l'aide du théorème de Krylov-Bogoliubov un théorème ergodique sur $(P_t^*)_{t\geq 0}$ ainsi que l'existence de mesures invariantes pour $(P_t)_{t\geq 0}$. On discute enfin en Sous-section IV.3.2 la question de la propriété de Feller forte, ce résultat étant basé un travail en cours, on donne essentiellement une idée de la marche à suivre en suivant la structure de [DD20].

Chapitre V : Équations paraboliques non-locales et quasilinéaires

• [BE24], I. Bailleul et H. Eulry, *Non-local quasilinear singular SPDEs*, preprint ArXiv:2405.18057 .

On s'intéresse dans ce chapitre à une version non-locale et quasilinéaire de l'équation PAM (parabolic Anderson model)

$$\partial_t u - A[f(u)]\Delta u = B[g(u)]\xi$$

posée sur le tore $\mathbb{T}^2$ de dimension 2, où A et B sont définis comme des opérateurs pseudo-différentiels. On traite ici le cas où l'action de A et B n'est pas nécessairement diagonale, mais plutôt donnée par un symbole $\sigma : \mathbb{T}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ supporté dans un cône en fréquences

$$\widehat{\sigma}(n,k) = 0 \quad \text{pour} \quad |n| > \mu(1+|k|).$$

Un exemple canonique est donné par des opérateurs de convolution ou plus généralement par des opérateurs à noyau. Sous une hypothèse de positivité de A, vérifiée par exemple si le noyau de A est une mesure positive $\nu(x,\cdot)$

$$A[u](\varphi) = \iint u(y)\nu(x,dy)\varphi(y)\,dx,$$

on est à même d'utiliser les outils du calcul paracontrôlé pour étudier l'équation renormalisée. On établit alors un résultat d'existence locale à l'équation

$$\partial_t u - A[f(u)]\Delta u = B[g(u)]\xi$$

pour des données initiales $u_0 \in \mathcal{C}^\alpha$ avec $\alpha \in (2/3, 1)$.

INTRODUCTION

Evolution problems driven by an input signal are omnipresent in mathematics as they present a very general framework in which one can model dynamical problems. In a finite dimensional setting, say for instance that the evolution in time of a system X is governed by non-linear functions f and g and driven by an input signal ξ, then we shall consider equations under the form

$$\frac{dX_t}{dt} = f(X_t, t) + g(X_t, t)\xi_t.$$

Now imagine the input data ξ is in fact a noise, for instance if we were to model the interaction between particles in a disordered medium, then ξ should represent a stochastic forcing term. As such, ξ lacks regularity and the equation above falls out of the scope of usual differential equations theory, this is the case of the white noise $\xi = \frac{dB}{dt}$ where B is a Brownian motion. The latter being nowhere differentiable, this requires a whole new theory known as stochastic calculus and Itô's theory. Stochatsic Partial Differential Equations (SPDEs) aim at giving an infinite dimensional counterpart to such models, where the system u now lives in a Banach space, where this time the non-linear terms F, G can also involve operators acting on the model

$$\partial_t u = F[u] + G[u]\xi.$$

The noise terms as ξ appearing in the equations still are really rough, typically being a white noise, making the resolution theory for such equations a real challenge. Furthermore, due to the irregularity of the functions in play, ill-posed terms may appear in the equations under the form of non-linear functionals of a distribution for instance. This is the case of the well-known Kardar–Parisi–Zhang (KPZ), Φ^4_3 or Parabolic Anderon Model (PAM) equations which we call Singular SPDEs. For these pathological equations, major progress has been observed in the past 20 years, providing new and powerful tools for their study. At the price of a so-called *renormalization procedure*, this allowed to handle such problems that were inaccessible before. We provide in this work only a few examples of the questions we are able to tackle with these tools at hand, and our goal is to emphasize the general mantra behind the treatment of singular SPDEs via paracontrolled calculus.

Singular SPDEs : the story so far

So, where did it all start ?

In a discrete setting, a noise ξ would be defined as a random function whose value at each point is a random variable. As a matter of convenience, assume that such a noise ξ

is centered and Gaussian, then it is fully determined by its correlation kernel

$$(x,y) \mapsto \mathbb{E}\Big[\xi(x)\xi(y)\Big].$$

In the continuous case, this may not even be a function, merely a distribution. Such is the case of the white noise, for which the covariance function is given by the Dirac mass. Namely, a white noise on $\mathbb{R}^d$ is a centered Gaussian process with

$$\mathbb{E}\Big[\xi(x)\xi(y)] = \delta_0(x-y),$$

in other words, ξ is indexed by smooth functions and its correlation kernel is given by the L^2 inner product

$$\mathbb{E}\Big[\langle \xi, \varphi\rangle\langle \xi, \psi\rangle\Big] = \int_{\mathbb{R}^d} \varphi(x)\psi(x)\,dx.$$

By nature, ξ is then a random object with very limited local regularity, for which we can prove that $\xi \in \mathcal{C}^{-\frac{d}{2}-\kappa}$ for any $\kappa > 0$, illustrating the general fact that *the more correlation, the more regularity.* Such a noise (be it space only, or space-time) appears naturally in classical models of SPDEs such as the KPZ equation

$$\partial_t u - \partial_x^2 u = \Big(\partial_x u\Big)^2 + \xi,$$

the Φ^4_d model

$$\partial_t u - \Delta u = -u - u^3 + \xi,$$

or the PAM equation

$$\partial_t u - \Delta u = u\xi.$$

Appart from the noise, these models all share a similar property : they involve a (possibly non-linear) operation that is ill-defined at first. Indeed, due to the poor regularity of ξ one cannot expect u to have much regularity, preventing the product operations above from making sense. For instance, $\partial_x u$ will be a distribution in the KPZ case, so that $\Big(\partial_x u\Big)^2$ does not make sense, same goes for u^3 in the Φ^4_d model when $d \geq 2$, and in the PAM case u is not expected to have enough regularity to make sense of the product $u\xi$. This is the *Singular* part of *Singular SPDEs.* The major consequence of this is that there is no hope of applying the usual analytical toolbox to tackle these equations, it is even unclear what we would call a solution to such a singular equation.

The lack of a robust analytical framework in which one could make sense of these equations has been resolved making the following observation : the usual functional spaces are not suitable for the analysis of singular SPDEs, so we need to study them in appropraite random spaces built on the noise itself. This is the idea behind Hairer's groundbreaking theory of *Regularity Structures*, introduced in [Hai14], where local regularity is measured in terms of noise-built reference objects. The parallel approach of *Paracontrolled Calculus* proposed by Gubinelli, Imkeller and Perkowski in [GIP15] relies on harmonic analysis tools to provide a global description of functions, putting the emphasis on their high frequency behavior where their regularity is measured. This was further enriched by Bailleul, Bernicot and Frey in [BB16], [BBF17] and [BB19] as well as by Mouzard in

[Mou22] to adapt these tools to the curved setting of a general manifold, constructing a high order paracontrolled calculus based on the heat semi-group. Altogether, the key point is to give a description of potential solutions u in terms of reference noise data, so that the singular problems only brings the reference objects into play. Dealing with this problem is called the *renormalization procedure* and relies solely on the probabilistic properties of the noise. Lously speaking, it consists in substracting an "infinite" correction term in the equation to kill the singular terms.
Take for instance the KPZ equation introduced in [KPZ86]

$$\partial_t u - \partial_x^2 u = \left(\partial_x u\right)^2 + \xi$$

on $\mathbb{R} \times \mathbb{T}$, where ξ is a space-time white noise and $\mathbb{T}$ the one-dimensional torus. The formal change of variable $u = \log(v)$, called the Hopf-Cole transform, yields the following stochastic heat equation on v

$$\partial_t v - \partial_x^2 v = v\xi.$$

The space variable being one-dimensional, this equation can be interpreted as an Itô equation

$$dv_t = \partial_x^2 v_t \, dt + v_t \, dW_t$$

where W is a cylindrical Brownian motion on $L^2(\mathbb{T})$. The latter has a unique mild solution thanks to the usual SPDE tools and we could arguee that $u = \log(v)$ is then solution to the KPZ equation. However, one has to use Itô's formula to properly find the equation solved by u and doing so, we end up with an infinite correction term, so that u rather satisfies

$$\partial_t u - \partial_x^2 u = \left(\partial_x u\right)^2 - \infty + \xi$$

at least formally (this can be made precise by replacing W by a mollified counterpart W_ε and then investigating the limit $\varepsilon \to 0$).
Both the theory of regularity structures and paracontrolled calculus have been fruitful ever since. We point out the works of Bailleul and Hoshino [BH21a] and [BH21b] where they offer a dictionnary to commute between regularity structures and paracontrolled calculus. Note also the work of Chauleur and Mouzard [CM23] where they study the logarithmic non-linear Schrödinger equation, being able to handle a non locally Lipschitz non-linearity.

The quasilinear setting for singular SPDEs has been subject to quite the development thanks to these tools, see for instance the works [BDH16; FG19; OW19; BHK23; BM23; OSSW23; LOT23]. In Chapter V we study a non-local version of the quasilinear PAM equation on the 2-dimensional torus. Using the tools from paracontrolled calculus and their interaction with pseudo-differential operators as studied in [BDFT23b], we follow the same idea as in [BDH16] to prove a local well-posedness result.

The Anderson Hamiltonian, a singular operator

In the case of a time independent multiplicative white noise as is the case for the linear PAM equation, one can even consider the singular product $u\xi$ to be part of the operator driving the dynamics. This yields the formal operator

$$\mathcal{H} = -\Delta + \xi$$

called the *Anderson Hamiltonian.* Because of the roughness of ξ, $\mathcal{H}$ does not fall into the scope of usual perturbation theory in dimension $d \geq 2$, but it turns out that the tools mentioned above are well-suited for the study of such a singular operator. It appeared first in [And58] as a way to model a mass concentration phenomenon for quantum systems under a random forcing, this is the so-called *Anderson localization* phenomenon. In a discrete setting, we refer to the book of König [Kön16] for a whole survey. As far as dimension 1 is concerned, the noise is regular enough for the operator to make sense directly, following for instance the construction of Fukushima and Nakao [FN77]. This does not mean it is any less interesting and the recent works from Dumaz and Labbé [DL20], [DL22] and [DL24] provide a very complete understanding of the properties of the operator in a 1-dimensional framework. In [DL24] they provide a whole survey on the spectral properties of $\mathcal{H}$ centered at some energy level E, depending on how E compares to the size of the segment on which the operator is constructed. In [DL22] they obtain a full understanding of the spectral properties of $\mathcal{H}$ on the whole real line, recovering the Anderson localization : the spectral measure of the operator is almost surely pure-point and eigenfunctions are exponentially localized.

Things get trickier in dimension 2 as the product with ξ is singular. The first construction amounts to Allez and Chouk in [AC15] where they are able to construct the operator on the torus $\mathbb{T}^2$. The idea was to introduce a space of paracontrolled functions as the domain of $\mathcal{H}$ and define it through a renormalization procedure. They are able to define $\mathcal{H}$ as an unbounded self-adjoint operator on $L^2(\mathbb{T}^2)$ with discrete spectrum, as well as get bounds on the eigenvalues in the large volume limit. Labbé achieved a similar result in [Lab19] using the tools from regularity structures, defining the operator in bounded domains in dimension $d \leq 3$. In [GUZ20], Gubinelli, Ugurcan and Zachhuber constructed the operator on $\mathbb{T}^2$ and $\mathbb{T}^3$ using a combination of the paracontrolled calculus and exponential transform to handle the 3-dimensional construction. Using the tools from high order paracontrolled calculus, Mouzard constructed in [Mou22] the operator in the general framework of a closed (compact, connected, boundaryless) Riemannian surface $\mathcal{M}$, proving it can be defined as a self-adjoint operator on $L^2(\mathcal{M})$ with random domain made of paracontrolled functions (as in [AC15], [Lab19] or [GUZ20]). He establishes that the operator has a pure point spectrum and its eigenvalues obey a Weyl-law. A simpler construction in dimension 2 and 3 via its quadratic form can also be found in [MO23]. Finally, in [BDM23], Bailleul, Dang and Mouzard provide a construction of $\mathcal{H}$, on a general surface, based on the construction of its resolvent. They are able to recover the properties from the aformentioned works as well as a fine understanding of the corresponding heat semigroup $e^{-t\mathcal{H}}$ and its kernel. They obtain properties really close to the ones one would expect for a usual perturbation of the Laplace operator making use of the Γ map that describes the domain of $\mathcal{H}$.

Having the operator and its properties at disposal, one can try to tranfer the usual results on $-\Delta$ to $\mathcal{H}$. For instance Schrödinger and wave equations on periodic domains have been investigated in [GUZ20]. Zacchuber in [Zac20] obtained Strichartz estimates for the Schrödinger equation driven by the Anderson Hamiltonian on $\mathbb{T}^2$, with Mouzard in [MZ22] they improve this result to the case of Riemannian surfaces, obtaining Strichartz estimates for $\mathcal{H}$ in the context of the Schrödinger and the wave equation. Debussche and Weber in [DW18], Debussche and Martin in [DM19], then Tzvetkov and Visciglia in [TV23b] and [TV23a] study the Schrödinger equation with a multiplicative spatial white noise using the so-called *exponential transform* originating from Hairer and Labbé's work, proving global well-posedness for well-prepared initial data. Together with Debussche and Liu in [DLTV23], they lift these results to the case of the full plane. In the case of elliptic equations, Zhang and Duan studied in [ZD23] the regularity of solutions to the stationary non-linear Schrödinger associated to $\mathcal{H}$.

As far as infinite volume in dimension 2 is concerned, the lack of compactness makes things even harder and one has to work with spatial weights to control the behavior at infinity. In [Uek], Ueki defines the operator on $\mathbb{R}^2$ using similar techniques as the ones from [Mou22]. Finally, in the very recent work [HL24], they are able to define the operator in the full space $\mathbb{R}^2$ and $\mathbb{R}^3$ by relying on the solution theory for PAM equation posed in the full space.

While we recall some basics on the construction of $\mathcal{H}$ on a 2-dimensional manifold in Chapter I, the main content of this chapter is to provide yet another result to convince that $\mathcal{H}$ can be seen as a perturbation of the Laplace operator. This is done by giving a quantitative comparison between Green functions of both $\mathcal{H}$ and $-\Delta+1$, this will be proven to be very useful in Chapter III and IV.

In Chapter II, we use some of the properties of $\mathcal{H}$ to study associated elliptic equations in a compact 2-dimensional setting. Unlike [ZD23], our focus in not on the regularity of the solutions, but rather on the adaptation of tools from non-linear analysis to the setting of random singular operators. We use the so-called *Mountain Pass* methods to ensure existence of multiple solutions to the stationnary Schrödinger equation in presence of a space white noise, as well as convex analysis tools to handle Hartree type non-linearities in a similar setting.

Invariant Gibbs measures and Quantum Field Theory

The so-called renormalization procedure on which relies the discussion above is not inherent to the framework of paracontrolled calculus or regularity structures, it has been a natural tool when investigating the questions of invariant measures and Quantum Field Theory (QFT). This is another theory in which singular SPDEs naturally appear, developped to provide a fine description of systems of fundamental particles interacting together.

In a discrete setting, let Λ be a finite set of labels, a configuration (or field) is then a real valued function $u:\Lambda\to\mathbb{R}$. Assume the system is governed by some energy

$$\mathcal{E}(u)=\frac{1}{2}\sum_{x\sim y}|u(x)-u(y)|^2+\frac{1}{2}\sum_{x}|u(x)|^2+\sum_{x}V(u(x))$$

with some potential energy V, where we write $x\sim y$ where $x,y\in\Lambda$ are "close" enough to interact with each other. Then the goal is to get an understanding of the corresponding

Gibbs measure

$$d\rho(u) \propto e^{-\mathcal{E}(u)} \prod_x du(x).$$

Note that the quadratic part of the energy can be considered together with $\Pi_x du(x)$ to obtain a new reference probability measure μ called the *Gaussian Free Field* (GFF) (or more precisely, a random variable with law μ is called a GFF) so that in the end

$$d\rho(u) \propto e^{-\sum\limits_x V(u(x))} d\mu(u).$$

This of course does not cause any issue in the finite setting, however it seems unclear how to make sense of such a measure in the continuous setting. Indeed, the potential energy term consists in a possibly non-linear function of a distribution u, which does not make any sense. As for the quadratic part however, there is some hope. Even though there is no such thing as Lebesgue measure in infinite dimensional spaces, one can still define μ as a Gaussian measure. The continuous counterpart of the discussion above yields the kinetic energy

$$\frac{1}{2}\langle u, (-\Delta + 1)u\rangle_{L^2}$$

which leads to the definition of a Gaussian Free Field in the continuous setting as a centered Gaussian random process, indexed by smooth functions, and with covariance operator $(-\Delta + 1)^{-1}$ (or equivalently with the Green function of $-\Delta + 1$ as covariance kernel). In a 2-dimensional compact domain for instance, the law $\mu^{-\Delta+1}$ of the GFF can also be defined as the law of the random series

$$u^\omega(x) = \sum_{n\geq 0} \frac{\gamma_n(\omega)}{\sqrt{1+\lambda_n^\Delta}} \varphi_n^\Delta(x)$$

where $(\lambda_n^\Delta, \varphi_n^\Delta)$ are the eigenvalues-eigenfunctions of Δ and γ_n are i.i.d. standard Gaussian random variables. This series can be shown to converge in $H^{-\kappa}$ for any $\kappa > 0$, but not in L^2, proving that generic elements in the support of $\mu^{-\Delta+1}$ are distributions, not functions. This raises the question of giving a proper definition of a density with respect to the law of the GFF whenever the potential V is non-linear. For instance when $V = \frac{1}{4}|\cdot|^4$, and the space is $\mathbb{T}^d$, this yields the well-known Φ^4 measure

$$d\rho(u) \propto e^{-\frac{1}{4}\int_{\mathbb{T}^d} |u(x)|^4\, dx} d\mu^{-\Delta+1}(u)$$

being the (formal) invariant measure to the Φ^4_d equation mentioned before

$$\partial_t u - \Delta u = -u - u^3 + \zeta$$

where ζ is a space-time white noise, and the equation is singular in dimension $d \geq 2$. The idea of seeing the Φ^4 measure as the invariant measure of the corresponding dynamics goes back to Parisi and Wu in [PW81] and is called *Stochastic Quantization*, translating the problem of understanding a measure into the study of a singular SPDE. Significant progress has been done ever since, especially since the 2003 work of Da Prato and Debussche [DD03] were they manage to obtain strong (in the probabilistic sense) solutions to the stochastic quantization equation on $\mathbb{T}^2$ with a polynomial non-linearity using the so-called *Da Prato-Debussche trick*. To name but a few works revolving arround the dynamical construction of the Φ^4 measure, Mourrat and Weber in [MW17b], Hairer and Mattingly in [HM17], Albeverio and Kusuoka in [AK20] constructed the dynamic and the associated Φ^4_3 measure on the 3-dimensional torus. See also the simple construction from Jagannath and Perkowski in [JP21]. As for the infinite volume case, Mourrat and Weber in [MW17a] proved the global well-posedness of the model in the plane, and in 3 dimensions, we refer to the works of Gubinelli and Hofmanová [GH19], Barashkov and Gubinelli in [BG20] and [BG23] or Albeverio and Kusuoka in [AK22]. While all these works take place in either the Euclidean space or the flat torus, it is also worth noting that some similar results exist in curved settings of Riemannian manifolds. For instance in the 2-dimensional case of a Riemannian closed surface, Burq, Thomann and Tzvetkov in [BTT18] are able to construct the polynomial Φ_2 measure and Oh, Robert, Tzvetkov and Wang in [ORTW20] with an exponential non-linearity. As far as the Φ^4 model on 3-dimensional manifolds is concerned, Bailleul, Dang, Ferdinand and Tô constructed the Φ^4_3 measure in the very recent works [BDFT23a] and [BDFT23b], and Bailleul in [Bai23] even proved uniqueness of the invariant measure.

In Chapter III, we aim at constructing the Anderson $P(\Phi)_2$ measure on a closed Riemannian surface for defocusing polynomial non-linearities. In other words, we study the Φ_2 equation, with polynomial non-linearity and perturbed by an independent space white noise ξ. This somehow amounts at studying the usual Φ_2 equation in a random and singular space environment. The corresponding dynamics is then driven by the Anderson operator $\mathcal{H}$, for which we obtain an invariant Gibbs measure described by a renormalized density with respect to the Anderson GFF measure. Our construction of the measure relies on the Boué-Dupuis formula as used in [BG20] and [BG23] to ensure the convergence of our renormalization procedure. Chapter IV is a continuation of this work, in which we aim at studying the properties of the measure following the steps of [MW17b] and [TW18] by constructing the measure dynamically through the equation. Note that in their very recent work [BDZ23], Barashkov, De Vecchi and Zacchuber also construct an Anderson Φ^4_2 measure in the context of the cubic wave equation.

Such invariant Gibbs measures can also be used to study the corresponding dispersive equations. This was the idea of Lebowitz, Rose and Speer seminal work [LRS88] to use statistical mechanics to get a better understanding of the equation. A few years latter this was used by Bourgain in [Bou94] on the circle and [Bou96] on $\mathbb{T}^2$ to study the non-linear Schrödinger equation (respectively with focusing and defocusing non-linearity)

$$i\partial_t u + \Delta u = \pm|u|^2 u$$

starting from irregular initial data. The problem with Schrödinger equation is that, unlike the heat equation, the corresponding semi-group does not provide any regularization. One has generally to resort to Strichartz inequalities to obtain estimates on the solution, which makes the set of initial data for which the problem is globally well-posed considerably smaller than it would be for the heat equation. The groundbreaking idea of Bourgain was, lacking good preserved quantities to ensure global well-posedness, to use invariant Gibbs measure as a way to globalize solutions. Instead of proving global well-posedness for any initial data, the idea is to state a globalization argument holding for initial data distributed according to the Gibbs measure to recover almost-sure global well-posedness. Dynamics leaving the Φ_d measure invariant have been studied in various contexts, for instance for the wave equations in [GKO18; GKOT22; ORT23; OT20; OTWZ22], Schrödinger equations [BL23; DNY19; OT18; Tzv08], and complex Ginzburg-Landau equations [DDF21; RZ; Tre19], to name but a few.

In [BTT18] and [OT20], a focus is given not on the global well-posedness of dynamics leaving the Gibbs measure invariant, as it can be a very difficult question, but rather as wether we can ensure existence of global solutions for rough initial data. After constructing an invariant Gibbs measure to the equation via a renomalization procedure, the idea is to use a compactness criterion to somehow provide a global solution having the invariant measure as law at each fixed time. The compactness argument takes the form of using successively Prohorov and Shorokhod's theorems after enhancing the spatial Gibbs measure into a space-time measure and prove tightness of the sequence of renormalized measures.

Perspectives

This thesis presents only a fraction of the many interesting problems that can be tackled within the framework of paracontrolled calculus. The results obtained here pave the way for further development. For instance, from Chapter II, the question of solitons for the Anderson Schrödinger equation posed in the full space is still an open question, there is some hope on the real line by using the analysis provided in [DL22], but the lack of compactness is a major issue.
A natural question after reading Chapter III and IV is the study of the finer properties of the Gibbs measure that is constructed therein. In the ongoing work on which is based Chapter IV we are able to prove the transition semigroup associated to the solution to the Anderson stochastic quantization equation has the strong Feller property, we also are able to prove that the Gibbs measure constructed before is actually the only invariant measure to the dynamic, as well as ergodicity of the system. This is described at the end of Chapter IV, in Section IV.3.2.
Having noted that the construction in Chapter III is quite robust, a follow-up question is to investigate the case of an analytic non-linearity (recovering for instance Liouville theory or sine-Gordon). Moreover, we strongly believe that the method used here can be applied as a sort of *black box* set of tools, working for any good enough operator.
In another work in progress, we investigate the dispersive counterpart of the equation studied in Chapter IV. We present this equation here as well as a schematic idea of how to tackle it. Given a spatial white noise ξ on the two dimensional torus $\mathbb{T}^2$, our focus is on the Schrödinger equation perturbed by ξ:

$$i\partial_t u + \Delta u = |u|^2 u + \xi u$$

on the torus $\mathbb{T}^2$, with initial data u_0 of limited regularity. Even without ξ, the nonlinear Schrödinger equation can be tricky due to the absence of regularization from the associated semigroup $e^{-it\Delta}$. Here, our concern is not the well-posedness of the equation, but rather whether we can ensure the existence of global solutions, even with irregular initial data. The primary tool in that case is constructing an invariant Gibbs measure for the equation, formally given by:

$$d\rho(u) \propto e^{-\frac{1}{2}\int_{\mathcal{M}} |\nabla u|^2 + \xi u^2 \, dx - \int_{\mathcal{M}} |u|^4 \, dx} du.$$

The construction of ρ follows the same lines as the one in Chapter III, though we now handle complex valued distributions. ρ is then defined as the total variation limit of a sequence of measures ρ_N with density with respect to the law of the Gaussian free field $\mu^{\mathcal{H}}$ of $\mathcal{H}$.
The idea is then to refine the sequence of measures $(\rho_N)_N$ into a sequence $(\nu_N)_N$ concerning space-time functions:

$$\nu_N = \rho_N \circ \Phi_N^{-1}$$

where Φ_N stands for the flow of the truncated equation preserving the invariance of ρ_N. We are then able to prove that the sequence $(\nu_N)_N$ is tight. By employing Prokhorov's and Skorokhod's theorems, we construct a sequence of functions $(u_N)_N$ distributed according to $(\nu_N)_N$, that converges to a global solution of the initial problem.

Overview of the thesis

Chapter I: Anderson Operator and Paracontrolled Calculus

In this chapter, we provide the analytical framework that will underpin the subsequent discussions. The concept of paracontrolled calculus originates from the Paley-Littlewood decomposition, applied to a distribution $u \in \mathcal{D}'(\mathbb{T}^d)$:

$$u = \sum_{j \geq -1} \Delta_j u$$

where the Paley blocks $\Delta_j u$ are centered around frequencies of size approximately 2^j. This allows us to express the product of two distributions u and v as follows:

$$\begin{aligned} uv &= \sum_{i<j-1} \Delta_i u \Delta_j v + \sum_{|i-j|\leq 1} \Delta_i u \Delta_j v + \sum_{j<i-1} \Delta_i u \Delta_j v \\ &= \mathsf{P}_u v + \mathsf{\Pi}(u,v) + \mathsf{P}_v u \end{aligned}$$

Here, $(\mathsf{P}, \mathsf{\Pi})$ denotes the paraproduct-resonance pair. While the paraproduct terms $\mathsf{P}_u v$ and $\mathsf{P}_v u$ are always well-defined, the resonant term $\mathsf{\Pi}(u,v)$ captures the singularity of the product uv. Consequently, addressing singular SPDEs involves providing an analytical interpretation of this a priori ill-defined term without assuming strong regularity on u or v.

Consider, for instance, the heat equation driven by spatial white noise ξ:

$$\partial_t u - \Delta u = f(u) + u\xi$$

defined on $\mathbb{R}_+ \times \mathbb{T}^2$. The term $u\xi$ is ill-defined due to the irregularity of ξ. However, utilizing the tools of paracontrolled calculus, we can give meaning to the equation by prescribing an a-priori structure on functions u as follows:

$$u = \mathsf{P}_u X + u^\sharp$$

Here, X depends solely on the noise ξ, while $u^\sharp$ represents a remainder term with improved regularity. After a purely probabilistic renormalization procedure, we are then able to operate within the framework of standard analytical tools.
Throughout this work, we focus on dynamics governed by the random operator

$$\mathcal{H} = -\Delta + \xi$$

referred to as the Anderson operator. Although $\mathcal{H}$ is somehow a perturbation of the Laplacian, usual perturbation techniques are unavailable due to the irregularity of ξ. Nevertheless, we manage to compare the operators at some level, for instance, we prove that the Green function of $\mathcal{H}$ and $-\Delta+1$ both have the same singularity on the diagonal (see Lemma I.9).

Green functions comparison

Let $4 \leq p < \infty$ and $0 < \delta < \frac{1}{2}$. Then $\mathcal{H}^{-1} - (1-\Delta)^{-1}$ is bounded from $W^{-\frac{2}{p}-\delta,p'}$ to $W^{\frac{2}{p}+\delta,p}$. In particular $(G^{\mathcal{H}} - G^{1-\Delta})(x,y) \in L^\infty_y W_x^{\frac{2}{p}+\delta,p}$, and $G^{\mathcal{H}}(x,y) \in L^\infty_x W_y^{\frac{2}{p}-\delta,p}$.

Do not be mislead however by this result, this does not allow to recover any kind of elliptic regularity from $\mathcal{H}$ as it regularizes in its own Sobolev spaces that do not embed in anything better than H^{1-}.
We also provide a quantitative comparison between symbols associated with each of these operators (see Lemma I.13)

Semi-classical multipliers

Let $\psi \in \mathcal{S}(\mathbb{R})$, $2 \leq p < \infty$ and $\frac{2}{p} - 1 < \sigma < \frac{2}{p} + 1$, and $0 < \kappa < \delta < 1 + \kappa$ and $\kappa - \delta \leq \beta \leq 2 + \kappa - \delta$ such that $0 \leq \sigma - \frac{2}{p} + \delta \leq 2$. Then there exists $C > 0$ such that for any $N \geq 1$ and $f \in H^\beta(\mathcal{M})$, it holds

$$\left\| \psi(N^{-2}\mathcal{H})f - \psi(N^{-2}\Delta)f \right\|_{W^{\sigma,p}} \leq C N^{\max(\sigma+\kappa-\frac{2}{p}-\beta,0)} \|f\|_{H^\beta}.$$

Moreover, if ψ is compactly supported away from 0, it holds

$$\left\| \psi(N^{-2}\mathcal{H})f - \psi(N^{-2}\Delta)f \right\|_{W^{\sigma,p}} \leq C N^{\sigma+\kappa-\frac{2}{p}-\beta} \|f\|_{H^\beta}.$$

Chapter II : Elliptic singular SPDEs

• [BER22], I. Bailleul, H. Eulry and T. Robert, *Variational methods for some singular stochastic elliptic PDEs*, to appear in *Annales de la Faculté des Sciences de Toulouse*.

After constructing the Anderson operator, our focus shifts to stationary equations driven by $\mathcal{H}$ posed on a compact surface $\mathcal{M}$. Using tools from nonlinear analysis, we delve into the study of a singular elliptic equation that cannot be tackled with conventional fixed-point methods like contraction or Schauder's theorem. Specifically, given a nonlinearity f and a potential $a \in L^p(\mathcal{M})$, $p > 1$, we consider the following equation:

$$(-\Delta + a)u = f(\cdot, u) + \xi u.$$

While standard fixed-point theorems ensure the existence of a solution under certain assumptions on a and f, this often requires some form of smallness assumption on a and integrability assumption on f. In the general case, characterizing solutions as critical points of a certain energy functional

$$\Phi(u) := \frac{1}{2}\|u\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} \left(\frac{1}{2}a(x)u(x)^2 - F\Big(x, u(x)\Big)\right) dx$$

we then resort to Mountain-Pass type compactness methods. This involves investigating perturbations of the operator $\mathcal{H}$ by integrable potentials, leading to the notion of a Kato class for the operator, which essentially ensures that $\mathcal{H} + a$ maintains the same spectral theory as $\mathcal{H}$.
We then prove that the elliptic problem

$$(\mathcal{H} + a)u = f(\cdot, u)$$

admits at least one solution provided a and f satisfy reasonable assumptions, including, for example, usual polynomial nonlinearity $f(x, u) = u|u|^\ell$. Under additional symmetry assumptions, e.g. when ℓ is even, we show that the equation admits infinitely many solutions, which stands in stark contrast with standard results on existence and uniqueness (see Corollary II.15).

Existence of stationnary solutions

For any non-null even integer ℓ and any potential $a \in L^p(\mathcal{M})$, with $p > 1$, the semilinear problem

$$\mathcal{H}u + au = u|u|^\ell$$

has infinitely many weak solutions.

With the spectral theory of $\mathcal{H} + a$ at our disposal, we also address equations that cannot be formulated as seeking critical points of energy functionals. This is for instance the case of the singular version of the Choquard-Pekar equation on the two-dimensional torus:

$$(-\Delta + a)u = \Big(w \star f(u)\Big)g(u) + \xi u,$$

where $\star$ denotes the convolution operation. Here, we employ Ghoussoub's self-dual functionals to establish the existence of a solution, taking the form of a point attaining the minimum of a strongly coercive self-dual functional under certain assumptions on the coefficients a, w, f, g of the equation (see Theorem II.17).

Stationnary solutions with Hartree-type non-linearity

Pick exponents $p \in [1, +\infty)$, $q \in (1, +\infty)$ and let the potential a be bounded and positive. Assume that the interaction kernel $w \in L^1(\mathbb{T}^2)$ is non-positive. Then the singular Choquard-Pekar equation

$$(\mathcal{H} + a)u = \Big(w \star |u|^p\Big)|u|^{q-2}u,$$

has a weak solution.

One of the main interests in studying such equations lies in obtaining the existence of stationary solutions for associated dispersive equations. For instance, considering a polynomial nonlinearity, our results ensure the existence of solitary waves (or solitons) for the focusing Schrödinger equation posed on $\mathbb{R} \times \mathcal{M}$:

$$i\partial_t v + \mathcal{H}v = |v|^{p-1}v$$

in the form $v(t, x) = e^{i\lambda t}u(x)$ where u satisfies

$$\mathcal{H}u + \lambda u = |u|^{p-1}u.$$

Similar reasoning applies in the case of a nonlocal nonlinearity as above:

$$i\partial_t v + \mathcal{H}v = \Big(w \star |v|^p\Big)|v|^{q-2}v$$

where we obtain v in the form $v(t, x) = e^{i\lambda t}u(x)$ with u satisfying the associated elliptic equation

$$\mathcal{H}u + \lambda u = \Big(w \star |u|^p\Big)|u|^{q-2}u.$$

Such a solution u then keeps a time-independent shape, modulated by space-independent oscillations.

Chapter III : Invariant Gibbs measures in singular environment

• [EMR24], H. Eulry, A. Mouzard and T. Robert, *Anderson stochastic quantization equation*, preprint ArXiv:2401.12742.

In this chapter, we delve into the quantization equation on a compact surface $\mathcal{M}$, driven by both a space white noise ξ and a space-time white noise ζ:

$$\partial_t u - \Delta u + \xi u + F'(u) = \sqrt{2}\zeta$$

where F is a defocusing polynomial nonlinearity. This equation presents us with two layers of randomness and singularity. On one hand, the $P(\Phi)_2$ model in the absence of ξ is already singular due to the poor regularity of ζ and necessitates a detailed investigation through a renormalization procedure of the equation. On the other hand, the Anderson operator $\mathcal{H} = -\Delta + \xi$ also exhibits singularity. Hence, we heavily rely on the resolution theory for $\mathcal{H}$, referring to the outcomes concerning the heat semigroup of the Anderson operator, its spectral theory, and its Green function, as discussed in Chapter I. The entire study of the equation rests upon the decomposition of ζ into a carefully chosen basis of $L^2(\mathcal{M})$:

$$\zeta(ds) = \sum_{n\geq 0} dB_n(s)\varphi_n^{\mathcal{H}}$$

where $(B_n)_{n\geq 0}$ is an i.i.d. family of real Brownian motions. This decomposition brings out the confrontation between the two sources of randomness: on one hand, the randomness of $\mathcal{H}$ through its eigenfunctions $\varphi_n^{\mathcal{H}}$, and on the other hand, the randomness of ζ through the coefficients B_n.
The primary goal of studying this dynamics is to propose a parabolic model for constructing an associated Gibbs measure:

$$d\rho(u) \propto e^{-\frac{1}{2}\int_{\mathcal{M}} |\nabla u|^2 + \xi u^2 \, dx + \int_{\mathcal{M}} F(u)\, dx}\, du$$

By separately considering the quadratic part, ρ is then defined as a measure with density with respect to the Gaussian free field $\mu^{\mathcal{H}}$ associated with $\mathcal{H}$, that is the law of the random series

$$u^\omega(x) = \sum_{n\geq 0} \frac{\gamma_n}{\sqrt{\lambda_n}}\varphi_n$$

where (λ_n, φ_n) are the eigenvalues-eigenfunctions of $\mathcal{H}$. Since $\mu^{\mathcal{H}}$ is supported in distribution spaces but not in $L^2(\mathcal{M})$, it is necessary to undergo a renormalization procedure to give meaning to the potential energy term $\int_{\mathcal{M}} F(u)\, dx$. The construction of ρ then heavily relies on the Boué-Dupuis formula and a passage to the limit in its renormalized counterparts ρ_N.
The Gibbs measure, being a candidate for an invariant measure to the equation, is constructed with density with respect to $\mu^{\mathcal{H}}$. We then aim to finely study the properties of the equation with initial data in the support of $\mu^{\mathcal{H}}$. This study is based on a clever decomposition of u following the idea of Da Prato and Debussche. Denoting Ψ as the stochastic convolution, i.e., the stationary solution of:

$$\partial_t \Psi + \mathcal{H}\Psi = \sqrt{2}\zeta$$

then $v = u - \stackrel{\bullet}{|}$ satisfies the nonlinear equation:

$$\partial_t v + \mathcal{H}v + F'(v + \stackrel{\bullet}{|}) = 0$$

from which we expect solutions to exhibit better regularity than u. The study of $\stackrel{\bullet}{|}$ requires a particular treatment to define its renormalized powers, given that $\stackrel{\bullet}{|}$ is a distribution a priori, we cannot directly define a nonlinear functional for it. This treatment is similar to that required for the potential energy term. Consequently, we obtain a local well-posedness result, as well as the convergence of the renormalized model to the limit equation (see Theorem III.2).

Deterministic local well-posedness

For $\varepsilon > 0$ small enough, $\mathbb{P}$-almost surely, for any $u_0 \in \mathcal{C}^{-\varepsilon}(\mathcal{M})$ and $T_0 > 0$, there exists $T \in (0; T_0 \wedge 1]$ such that for any $N \in \mathbb{N}^*$, (III.9) admits a unique solution $u_N \in C([0;T]; \mathcal{C}^{-\varepsilon}(\mathcal{M}))$. Moreover u_N converges almost surely to some u which is the unique solution to

$$\begin{cases} \partial_t u + \mathcal{H}u + f^{\diamond}(u) = \sqrt{2}\zeta. \\ u(0) = u_0. \end{cases}$$

Finally, we prove that the measure ρ is invariant under the dynamics and that solutions exist globally for initial data distributed according to this measure. The key argument for globalizing, in the absence of a conservation law, is to use the invariant measure itself as an invariant. Consequently, we obtain an expectation bound on the equation flow

$$\int_{\mathcal{C}^{-\varepsilon} \times \Omega} \sup_{0 \le t \le T \wedge T^*} \|\Phi(u_0, \omega)(t)\|_{\mathcal{C}^{-\varepsilon}} \, d\rho(u_0) \, d\mathbb{P}(\omega) \le C_T,$$

which is enough to ensure global well-posedness almost surely for initial conditions distributed according to ρ (see Theorem III.3)

Probabilistic global well-posedness and measure invariance

If u_0 is distributed according to ρ, then almost surely, for any $T > 0$ and $N \in \mathbb{N}^*$, the truncated equation (III.9) admits a unique solution on $[0, T]$. Moreover u_N converges in $C([0;T]; \mathcal{C}^{-\varepsilon}(\mathcal{M}))$ to the unique solution u to the fully renormalized equation, and the law of $u(t)$ does not depend on t and is given by ρ.

Chapter IV : Ergodicity for the Anderson Φ^4_2 measure

- A. Debussche, H. Eulry and A. Mouzard, *Ergodicity for the Anderson Φ^4_2 measure*, in preparation.

This chapter is mostly a continuation of the previous one. We consider the Anderson Φ^4_2 model on the 2 dimensional torus $\mathbb{T}^2$

$$\partial_t u + \mathcal{H}u + u^3 = \sqrt{2}\zeta$$

with initial data u_0. Local well-posedness theory for this equation falls under the scope of Chapter III, which provides us with a way to globalize solutions using the invariant measure constructed therein. We investigate the question of deterministic global well-posedness as well as probabilistic properties of the so-called *Anderson Φ^4_2 measure*, the final goal being to provide a construction of the previous measure via the dynamic. Writting $u = \text{¡} + v$ as before, we can get rid of the white noise term at the price of having a noise forcing in the non-linear term of the equation

$$\partial_t v + \mathcal{H}v + (v + \text{¡})^3 = 0.$$

This time, the noise term ¡ and its renormalized powers will be constructed so that v starts from the same initial condition as u does, namely ¡ will solve the linear equation

$$\partial_t \text{¡} + \mathcal{H}\text{¡} = \sqrt{2}\zeta\,, \quad \text{¡}_{t=0} = 0.$$

The trick to establish global well-posedness then usualy relies on obtaining energy bounds on v to satisfy a blow-up criterion. However, because of the singular nature of $\mathcal{H}$, it is unclear how to obtain L^p bounds on v by testing the equation against some power of v. This can be solved by using the Γ mapping used to define the Anderson Hamiltonian. As v solves the equation driven by $\mathcal{H}$, v has a paracontrolled expansion $v = v \olessthan X + w = \Gamma w$ where X is such that $\Delta X = \xi$. That is, Γw is defined as the fixed point solution to

$$\Gamma w = \Gamma w \olessthan X + w,$$

and is the inverse mapping to $v \mapsto v - v \olessthan X$. The key in the analysis is that w has much better regularity than v and that w solves an equation driven by the usual Laplace operator

$$\partial_t w - \Delta w + w^3 + R(w) + Q(w) = 0$$

where $R = \Gamma^{-1}\mathcal{H}\Gamma + \Delta$ is a linear operator losing 1^+ derivatives. The fact that it loses regularity comes from the irregularity of $\mathcal{H}$ and cannot be avoided. Q on the other hand gathers the non-linear contributions coming from the cubic term, this is where the noise data denoted by $\mathbf{z}$ will come into play. Namely we expand the cubic power $(v + \text{¡})^3$ as

$$\begin{aligned}\Gamma^{-1}\Big((v+\text{¡})^3\Big) &= \Big((\Gamma w)\olessthan X + w + \text{¡}\Big)^3 - \Big(\Big((\Gamma w)\olessthan X + w + \text{¡}\Big)^3\Big)\olessthan X \\ &= w^3 + Q(w),\end{aligned}$$

where Q only gathers terms with a lower homogeneity or involving the regularizing operator $\cdot \circledcirc X$. Up to a truncation of X, we are able to treat Q as a perturbative term in the equation. Also, while this does not play any role in the local well-posedness theory, we heavily rely on the fact that the cubic term u^3 comes with a positive sign in the equation. Testing the equation against w^{3p-3}, we obtain an a-priori estimate under the form of Proposition IV.12

A priori estimate

For any $0 < s < t < T$, $p > \frac{4}{3\varepsilon}$ even integer and $\varepsilon > 0$ small enough, then

$$\int_s^t \left(\|w_r\|_{L^{3p-2}}^{3p-2} + \|w_r\|_{B^{1+4\varepsilon}_{p,\infty}}^{\frac{3p-2}{3}} \right)^{\frac{3p}{3p-2}} dr \lesssim 1 + \|w_s\|_{L^{3p-2}}^{3p-2} + \|w_s\|_{B^{1+4\varepsilon}_{p,\infty}}^{\frac{3p-2}{3}}.$$

The implicit constant in this estimate does not depend on the initial data, and on the randomness outside of the time interval $[0,t]$. This allows to obtain a global estimate for w, uniformly in the initial condition. In view of the blow-up criterion established in Chapter III, this proves that solutions to Anderson Φ^4_2 equation are global and even get a bound on the moments of the solution (see Proposition IV.16 and Corollary IV.18).

Coming down from infinity

For any even integer $p > \frac{4}{3\varepsilon}$ and $\varepsilon > 0$ small enough

$$\forall 0 < t < T\,, \ (1 \wedge \sqrt{t}) \|w_t\|_{L^{3p-2}} \leq \tilde{K}_t$$

for some constant that depends on the noise only on $[0,t]$. As such, the solution to (IV.6) starting from from $u_0 \in \mathcal{C}^{-\varepsilon}$ is global in time. Moreover, the the right-hand side does not depend on u_0 and the following boud on moments holds

$$\sup_{u_0 \in \mathcal{C}^{\varepsilon}} \sup_{t \geq 0} \left(1 \wedge t^{\frac{3p-2}{2}}\right) \mathbb{E}\left[\|u(t;u_0)\|_{\mathcal{C}^{-\varepsilon}}^{3p-2} \right] < +\infty.$$

Since ζ is a white noise, we can expect nice properties on the solution u as a $\mathcal{C}^{-\varepsilon}$ valued process. Define its transition semigroup

$$P_t \Phi(u_0) = \mathbb{E}\left[\Phi\Big(u(t;u_0)\Big) \right]$$

for any $\Phi \in C_b(\mathcal{C}^{-\varepsilon})$ and the natural filtration $\mathcal{F}_t$ generated by ζ. Making full use of the Gaussian properties of ⫯, we can easily prove that $t \mapsto u(t;\cdot)$ is a Markov process satisfying the Feller property. Krylov-Bogoliubov theorem then ensures existence of invariant measures at the price of working in a slightly smaller space $\mathcal{C}^{-\varepsilon_0}$ with $\varepsilon_0 = \frac{\varepsilon}{2}$ (see Proposition IV.21).

Existence of invariant measures

The semigroup $(P_t)_{t\geq 0}$ is a Markov semigroup having the Feller property on $\mathcal{C}^{-\varepsilon_0}$. Moreover, if $u_0 \in \mathcal{C}^{-\varepsilon_0}(\mathbb{T}^2)$, there is a sequence of times $0 < t_0 < t_1 < \cdots \to +\infty$ and a probability measure μ_{u_0} on $\mathcal{C}^{-\varepsilon_0}(\mathbb{T}^2)$ such that

$$\frac{1}{t_k}\int_0^{t_k} P_r^*\delta_{u_0}\,dr \to \mu_{u_0}$$

where the convergence holds weakly. As such, μ_{u_0} is an invariant measure for $(P_t)_{t\geq 0}$ acting on $C_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$.

Finally in Subection IV.3.2, we discuss the strong Feller property. As it is still work in progress, we only give a schematic idea of the proof following the lines of [DD20], putting the emphasis on the tools needed.

Chapter V : Quasilinear non-local Parabolic Anderson Model (PAM)

• [BE24], I. Bailleul and H. Eulry, *Non-local quasilinear singular SPDEs*, preprint ArXiv:2405.18057 .

In this chapter, we focus on a non-local and quasi-linear version of the Parabolic Anderson Model (PAM) equation:

$$\partial_t u - A[f(u)]\Delta u = B[g(u)]\xi$$

posed on the torus $\mathbb{T}^2$ of dimension 2, where A and B are defined as pseudodifferential operators. We address the case where the action of A and B is not necessarily diagonal, but rather given by a symbol $\sigma : \mathbb{T}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ supported in a frequency cone:

$$\widehat{\sigma}(n,k) = 0 \quad \text{for} \quad |n| > \mu(1+|k|).$$

A canonical example is provided by convolution operators or more generally by kernel operators. Under a positivity assumption on A, verified for instance if the kernel of A is a positive measure $\nu(x,\cdot)$:

$$A[u](\varphi) = \iint u(y)\nu(x,dy)\varphi(y)\,dx,$$

we are able to use tools from paracontrolled calculus to study the renormalized equation. We then establish a local existence result for the equation:

$$\partial_t u - A[f(u)]\Delta u = B[g(u)]\xi$$

for initial data $u_0 \in \mathcal{C}^\alpha$ with $\alpha \in (2/3, 1)$ (see Theorem V.1).

Quasilinear non-local gPAM

For $2/3 < \beta < \alpha < 1$ and $s \leq 0$, take two symbols $a, b \in \Sigma_\alpha^s$ and assume that the operator A preserves positivity. Take further $f, g \in C_b^3(\mathbb{R})$, then

$$\partial_t u - A[f(u)]\Delta u = B[g(u)]\,\xi$$

has a well-defined formulation in a space of paracontrolled functions, where it has a unique solution (u, u'). Moreover there exists a positive random time T such the solution of the renormalized equation starting from $u_0 \in \mathcal{C}^\alpha(\mathbb{T}^2)$

$$\partial_t u^\varepsilon - A\Big[f(u^\varepsilon)\Big]\Delta u^\varepsilon = B\Big[g(u^\varepsilon)\Big]\,\xi^\varepsilon + c_a(\varepsilon)\left(\frac{B\big[g(u^\varepsilon)\big]}{A\big[f(u^\varepsilon)\big]}\right)^2 f'(u^\varepsilon) - c_b(\varepsilon)\frac{B\big[g(u^\varepsilon)\big]}{A\big[f(u^\varepsilon)\big]}\,g'(u^\varepsilon)$$

converges in probability in $\mathscr{C}_T^\alpha$ to u.

Notations

We collect here most of the common notations that will be used throughout this work.

Symbol	Meaning
$\mathbb{T}^d$	d-dimensional torus
$\mathcal{M}$	closed Riemannian surface
ξ	space white-noise
ζ	space-time white noise
$B^{\alpha}_{p,q}$	Besov space
$\mathcal{C}^{\alpha}$	Besov-Hölder space
$\mathscr{C}^{\alpha}_T$	parabolic time-space Hölder space
$\mathcal{H}$	the Anderson operator $-\Delta+\xi$
$W^{\sigma,p}$	Sobolev space
H^{α}	L^2-based Sobolev space
$\mathcal{D}^{\sigma}$	$\mathcal{H}$-based Sobolev space
$(\mathsf{P},\mathsf{\Pi})$	paraproduct-resonance pair
$(\otimes,\circleddash)$	tuned paraproduct-resonance pair
$\mathbf{P}_N$	regularizing operator $e^{N^{-2}\Delta}$
$X^{\diamond k}$	Wick power of X
$\int_{\mathcal{M}} f$ or $\int_{\mathcal{M}} f(x)\,dx$	integral with respect to the volume measure
$\langle\cdot,\cdot\rangle$	$L^2(\mathcal{M})$ inner product/duality bracket
$\langle k\rangle := (1+\lvert k\rvert^2)^{\frac{1}{2}}$	Japanese bracket of k
$\mathbb{E}_{\mu}$	expectation with respect to μ
$\lesssim_p$	bounded up to a constant depending on p
$\asymp$	bounded from above and below up to a constant
$\propto$	proportional

I The Anderson Hamiltonian

Contents

I.1 – Survival kit on paracontrolled calculus

We provide a short summary of the idea behind paracontrolled calculus as well as usual and most useful results, postponning the most technical ones to Appendix A.3. Proofs as well as use of these tools in a very instructive framework can be found in the seminal work of Gubinelli, Imkeller and Perkowski [GIP15]. The notion of paraproduct goes back to Bony [Bon81] based on the Littlewood-Paley decomposition and can be extended to manifolds, following for example [Mou22]. The main idea is one can decompose a product as

$$uv = \mathsf{P}_u v + \mathsf{\Pi}(u,v) + \mathsf{P}_v u$$

for any distribution $u \in \mathcal{C}^\alpha$ and $v \in \mathcal{C}^\beta$ where the resonant term $\mathsf{\Pi}(u,v)$ is well-defined only if $\alpha + \beta > 0$, this is Young condition. Precisely, let χ and ρ be smooth compactly supported radial functions on $\mathbb{R}^d$. Assume χ is supported in a ball and ρ is supported in an annulus. Set $\rho_j(\cdot) := \rho(2^{-j}\cdot)$ for $j \geq 0$ and $\rho_{-1} := \chi$. One can find χ and ρ such that

$$\sum_{j\geq -1} \rho_j \equiv 1 \quad \text{and} \quad \rho_j\rho_i \equiv 0 \text{ for } |i-j| \geq 0$$

The i-th Littlewood-Paley projector is defined as $\Delta_j f := \mathcal{F}^{-1}(\rho_j \widehat{f}\,)$. In those terms one has the formal decomposition

$$\begin{aligned} uv &= \sum_{i<j-1} (\Delta_i u)(\Delta_j v) + \sum_{|i-j|\leq 1} (\Delta_i u)(\Delta_j v) + \sum_{i>j+1} (\Delta_i u)(\Delta_j v) \\ &=: \mathsf{P}_u v + \mathsf{\Pi}(u,v) + \mathsf{P}_v u \end{aligned}$$

The operator P is called paraproduct operator and the operator $\mathsf{\Pi}$ the resonant operator. One can think of $\mathsf{P}_u v$ as a function or distribution that globally *looks like* v, but with lower frequencies modulated by u. This decomposition allows for a precise track of regularity with the continuity result

Proposition I.1 –

$$\|\mathsf{P}_u v\|_{B^{\alpha\wedge 0+\beta}_{p,q}} \lesssim \|u\|_{B^{\alpha\wedge 0}_{p_1,q}} \|v\|_{B^{\beta}_{p_2,q}}$$

for any $\alpha, \beta \in \mathbb{R}$ and $1 \leq p_1, p_2, p, q \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and

$$\|\mathsf{\Pi}(u,v)\|_{B^{\alpha+\beta}_{p,q}} \lesssim \|u\|_{B^{\alpha}_{p_1,q}} \|v\|_{B^{\beta}_{p_2,q}}$$

for any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$.

In view of the regularity condition $\alpha + \beta > 0$, it is clear that the ill-defined part of a product comes from its resonant term. This can however be circumvented for functions having themselves a paracontrolled expansion. This is the purpose of the following operator, first introduced in [GIP15], see Lemma 2.4 therein. Define the operator

$$\mathsf{C} : (u, v, w) \mapsto \mathsf{\Pi}(\mathsf{P}_u v, w) - u\, \mathsf{\Pi}(v, w)$$

on the set of smooth functions on $\mathbb{T}^2$.

Proposition I.2 – *Pick some regularity exponents $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}$ such that $\beta+\gamma < 0$ and $\alpha+\beta+\gamma > 0$. The operator C has a unique extension into a bounded trilinear operator from $\mathcal{C}^\alpha(\mathbb{T}^2) \times \mathcal{C}^\beta(\mathbb{T}^2) \times \mathcal{C}^\gamma(\mathbb{T}^2)$ to $\mathcal{C}^{\alpha+\beta+\gamma}(\mathbb{T}^2)$.*

With these results at hand, we could already almost tackle linear singular SPDEs such as the Parabolic Anderson Model (PAM)

$$\partial_t u - \Delta u = u\xi$$

on the 2-dimensional torus $\mathbb{T}^2$. The idea is then to work in a random subspace of functions having a paracontrolled expansion

$$u = \mathsf{P}_{u'} X + u^\sharp \tag{I.1}$$

where X depends only on the noise ξ and the remainder $u^\sharp$ has better regularity than u itself. We then run a fixed point argument on the pair $(u', u^\sharp)$ rather than u. Note that the corrector C is defined to get around singular products, but other commutators and correctors might be needed to investigate well-defined products that may not be written in a good form for the analysis. A thorough study of these commutators in a very general framework can be found in the works of Bailleul and Bernicot [BB16] and [BB19]. We only retain the following result on the merging operator that can be found in Theorem 3.6 of [BB16].

Proposition I.3 – *For $0 < \beta \leq \alpha < 1$ and $h_1 \in \mathcal{C}^\alpha(\mathbb{T}^2), h_2 \in C^\beta(\mathbb{T}^2)$ and $h_3 \in \mathcal{C}^\alpha(\mathbb{T}^2)$ one has*

$$\left\|\mathsf{P}_{h_1}\Big(\mathsf{P}_{h_2} h_3\Big) - \mathsf{P}_{h_1 h_2} h_3\right\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|h_1\|_{\mathcal{C}^\alpha} \|h_2\|_{\mathcal{C}^\beta} \|h_3\|_{\mathcal{C}^\alpha}.$$

Note also that similar construction and estimates can be obtained in the curved setting of a general closed surface using paracontrolled calculus based on the heat semi-group, we refer to [BB16] and [Mou22] for further details on this specific setting. The idea there is to replace the Littlewood-Paley decomposition by the following Calderon's reproducing formula for the heat semi-group

$$u = e^{-\Delta} u + \int_0^1 (-t\Delta) e^{t\Delta} u \frac{dt}{t}.$$

Up to a refinment of said formula, one is able to define analogous paraproduct-resonant pairs, thinking of $t^{-\frac{1}{2}}$ as the frequency parameter 2^n from the Fourier setting.
As for the non-linear case, we need to further investigate the action of non-linear functions on paraproducts, this is done via Bony's paralinearization lemma. For $f \in C_b^2(\mathbb{R})$ and $u \in \mathcal{C}^\alpha(\mathbb{T}^2)$ with $\alpha > 0$ set

$$R_f(u) := f \circ u - \mathsf{P}_{f' \circ u} u.$$

then we have the paralinearization estimate (see for instance Theorem 5.2.5 in [Mét08] or Theorem 2.92 in [BCD11]).

Proposition I.4 – *For $f \in C_b^2(\mathbb{R})$ and $u \in \mathcal{C}^\alpha(\mathbb{T}^2)$ with $\alpha > 0$ one has*

$$\|R_f(u)\|_{\mathcal{C}^{2\alpha}} \lesssim \|f\|_{C_b^2}(1 + \|u\|_{\mathcal{C}^\alpha}^2)$$

Moreover we have a local-Lipschitz estimate

$$\Big\| f \circ u - f \circ v \Big\|_{\mathcal{C}^\alpha} \le \|f\|_{C_b^2}\Big(1 + \|u\|_{\mathcal{C}^\alpha}\Big)\|u - v\|_{\mathcal{C}^\alpha}$$

This can be further improved as in Lemma C.1. of [GIP15] to the special case of functions having an expansion with a smoother remainder term as in (I.1).

Proposition I.5 – *Pick $0 < \beta \le \alpha < 1$ such that $\alpha + \beta > 1$. For $u \in \mathcal{C}^\alpha$ and $v \in \mathcal{C}^{\alpha+\beta}$, for $f \in C_b^3(\mathbb{R})$, one has*

$$\Big\| f(u+v) - \mathsf{P}_{f'(u+v)}(u+v) \Big\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{C_b^3}\Big(1 + \|u\|_{\mathcal{C}^\alpha}^{1+\beta/\alpha} + \|v\|_{L^\infty}^2\Big)(1 + \|v\|_{\mathcal{C}^{\alpha+\beta}})$$

I.2 – A singular operator: the Anderson Hamiltonian

In this section, we explain the construction of the Anderson Hamiltonian and state its properties used throughout this work. It was first constructed by Allez and Chouk [AC15] on the two-dimensional torus and then extended to various framework, see [GUZ20; Lab19; Mou22] and references therein. It is the Schrödinger operator

$$\mathcal{H} = -\Delta + \xi \tag{I.2}$$

with potential the spatial white noise on $\mathcal{M}$. As explained, the white noise is a random distribution of negative Hölder regularity and belongs almost surely to $\mathcal{C}^{-1-\kappa}$ for any $\kappa > 0$. In one dimension, the white noise is the derivative of the Brownian motion and was first constructed by Paley and Zygmund [PZ30; PZ32]. In this case, $\xi \in \mathcal{C}^{-\frac{1}{2}-\kappa}$ and the quadratic form $\langle(-\partial_x^2+\xi)u, u\rangle$ is well-defined with domain $H^1(\mathbb{T})$, see Fukushima and Nakao [FN77]. Having a precise description of its domain and its large volume limit was the subject of recent work by Dumaz and Labbé, see [DL24] and references therein. In two dimensions, this operator is singular in the sense that a renormalisation procedure is needed to make sense of $\mathcal{H}$. This is hinted by the fact that the product $u\xi$ is singular for $u \in H^1(\mathcal{M})$ hence the quadratic form is not well-defined. Following the recent progress in singular SPDEs, the operator can be defined as the limit of the renormalised operator

$$\mathcal{H} = \lim_{\varepsilon \to 0} (-\Delta + \xi_\varepsilon - c_\varepsilon)$$

with ξ_ε a mollification of the noise and c_ε a logarithmic diverging quantity. Following the idea of paracontrolled calculus from Gubinelli, Imkeller and Perkowski [GIP15], one can consider function u paracontrolled by a functionnal of the noise as domain for the operator or its quadratic form.

I.2.1 Construction of the operator

Consider the space

$$\mathcal{D}^\sigma := \{u \in L^2(\mathcal{M}) \ ; \ u - \mathsf{P}_u X_1 \in H^\sigma\}$$

with X_1 solution to $\Delta X_1 = \xi$ and P a paraproduct for $\sigma \geq 1$. The idea motivating the introduction of $\mathcal{D}^\sigma$ lies in the algebraic cancellation

$$\begin{aligned} -\Delta u + u\xi &= -\Delta \mathsf{P}_u X_1 - \Delta u^\sharp + \mathsf{P}_u \xi + \mathsf{\Pi}(u,\xi) + \mathsf{P}_\xi u \\ &= \mathsf{P}_u(-\Delta X + \xi) - \Delta u^\sharp + [\Delta, \mathsf{P}_u]X_1 + \mathsf{\Pi}(u,\xi) + \mathsf{P}_\xi u \\ &= -\Delta u^\sharp + [\Delta, \mathsf{P}_u]X_1 + \mathsf{\Pi}(u,\xi) + \mathsf{P}_\xi u \end{aligned}$$

where $u = \mathsf{P}_u X_1 + u^\sharp \in \mathcal{D}^\sigma$. This shows that introducing roughness in u via the paracontrolled expansion cancels exactly the most irregular term in $\mathcal{H}u$. However, this introduces a new singular product of distribution in $\mathsf{\Pi}(u,\xi)$. Indeed, $\xi \in \mathcal{C}^{-1-\kappa}$ thus $X_1 \in \mathcal{C}^{1-\kappa}$ hence the sum of the regularity exponent $-1-\kappa+1-\kappa = -2\kappa < 0$ barely fails Young condition. This is the singular nature of the operator and [GIP15] introduced the corrector C to write

$$\mathsf{\Pi}(u,\xi) = \mathsf{\Pi}(\mathsf{P}_u X_1, \xi) + \mathsf{\Pi}(u^\sharp, \xi) = u\mathsf{\Pi}(X_1,\xi) + \mathsf{C}(u, X_1, \xi) + \mathsf{\Pi}(u^\sharp, \xi)$$

for $u \in \mathcal{D}^\sigma$. For $\sigma > 1$, the resonant product $\mathsf{\Pi}(u^\sharp, \xi)$ is well-defined and one can prove that C is a continuous trilinear operator from $H^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to $H^{\alpha+\beta+\gamma}$ for $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\beta + \gamma < 0 < \alpha + \beta + \gamma < 1.$$

The condition $\beta + \gamma < 0$ corresponds exactly to the singular nature of the product $X_1\xi$ hence $\mathsf{C}(u, X_1, \xi)$ is well-defined as a function as soon as u is regular enough. This does not change that the resonant product $\mathsf{\Pi}(X_1, \xi)$ is singular hence the need for a renormalisation procedure, this can be done with the Wick product

$$\mathsf{\Pi}(X_1,\xi) = \lim_{\varepsilon \to 0} \Big(\mathsf{\Pi}(X_{1,\varepsilon}, \xi_\varepsilon) - \mathbb{E}\big[\mathsf{\Pi}(X_{1,\varepsilon}, \xi_\varepsilon)\big] \Big)$$

with ξ_ε a mollification of the noise and $\Delta X_{1,\varepsilon} = \xi_\varepsilon$. After this procedure, one obtains that $\mathcal{H}$ is well-defined on $\mathcal{D}^\sigma$ for $\sigma > 1$ and

$$\|\mathcal{H}u + \Delta u^\sharp\|_{H^{-\kappa}} \lesssim \|u\|_{\mathcal{D}^{1+\varepsilon}}$$

for any $\kappa, \varepsilon > 0$. In particular, the form quadratic $\langle \mathcal{H}u, u\rangle$ is well-defined and one can prove

$$\langle \mathcal{H}u, u\rangle + c\|u\|_{L^2}^2 \geq \frac{1}{2}\|u\|_{\mathcal{D}^1}^2 \geq 0$$

for a random constant $c > 0$ using almost duality, see for example [Mou22, Proposition 2.9]. At this point, one can prove that the quadratic form $\langle \mathcal{H}u, u\rangle$ with domain $\mathcal{D}^1$ is closed, symmetric, continuous and bounded from below thus its associated operator is self-adjoint. One can then obtain a second order paracontrolled expansion of the domain as

$$\mathcal{D}^2(\mathcal{H}) = \{u \in L^2(\mathcal{M}) \; ; \; u - \mathsf{P}_u X \in H^2\}$$

with $X = X_1 + X_2$ and X_2 solution to $\Delta X_2 = \Pi(X_1, \xi) + \mathsf{P}_\xi X_1$, see [Mou22] for this construction. The operator $\mathcal{H}$ has a compact resolvent with discrete spectrum $\lambda_0 < \lambda_1 \leq \ldots$ with a basis of eigenfunctions $(\varphi_n)_{n\geq 1}$, see [MO23] for a simple argument of the spectral gap. In the following, we suppose that the operator is positive considering for example that we include in the renormalisation the shift by the random constant $-\lambda_0 + 1$. Moreover, it was proved in [Mou22] that the sequence of eigenvalues satisfies the same Weyl law (A.1) as that of the Laplace-Beltrami operator, namely

$$\frac{\lambda_n}{n} \sim \frac{|\mathcal{M}|}{4\pi}. \tag{I.3}$$

The form domain $\mathcal{D}(\sqrt{H}) = \mathcal{D}^1$ is the reciproque image of H^1 by the application

$$\Phi(u) = u - \mathsf{P}_u X$$

which is continuous from H^σ to itself for any $\sigma < 1$. Up to a random truncation of the noise which does not change the regularity, one can prove that $\Phi : H^\sigma \to H^\sigma$ is invertible as a perturbation of the identity, we denote by Γ its inverse. It can be seen as the implicit solution to

$$\Gamma u^\sharp = \mathsf{P}_{\Gamma u^\sharp} X + u^\sharp \tag{I.4}$$

for $u^\sharp \in L^2(\mathcal{M})$.

In fact, due to X having $\mathcal{C}^{1-\kappa}$ regularity for any $\kappa > 0$, one can follow the same lines as in the proof of proposition 2.4 in [Mou22] to obtain the following crucial fact.

Lemma I.6 – *For any $\sigma \in (-1, 1)$, there exists a random truncation of the noise depending only on σ such that*

$$\Phi : \mathcal{C}^\sigma \to \mathcal{C}^\sigma$$

is invertible. As such, for any $p > 16$ and $\alpha \in (-\frac{1}{8}, \frac{1}{2})$, there exists positive constants $c, C > 0$ such that

$$c\|u\|_{B^\alpha_{p,\infty}} \leq \|u^\sharp\|_{B^\alpha_{p,\infty}} \leq C\|u\|_{B^\alpha_{p,\infty}}$$

Proof – Proposition I.1 ensures that for any $u \in \mathcal{C}^\sigma$ and $\kappa > 0$ such that $|\sigma| < 1-\kappa$

$$\|\mathsf{P}_u X\|_{\mathcal{C}^\sigma} \lesssim \|u\|_{\mathcal{C}^\sigma}\|X\|_{\mathcal{C}^{1-\kappa}}$$

where the implicit constant depends only on σ, proving the continuity of $\Phi : \mathcal{C}^\sigma \to \mathcal{C}^\sigma$. As in Proposition 2.4 in [Mou22], up to a truncation of X, the implicit constant above can be made strictly less than 1. As such Φ is a small perturbation of the identity, and thus is invertible. Set $\sigma = -\frac{1}{4}$, since $\alpha - \frac{2}{p} > -\frac{1}{4}$ and in view of the Besov embedding $B^\alpha_{p,\infty} \hookrightarrow \mathcal{C}^{\alpha-\frac{2}{p}} \hookrightarrow \mathcal{C}^{-\frac{1}{4}}$ from Lemma A.3 we have

$$\begin{aligned}\|\mathsf{P}_u X\|_{B^\alpha_{p,\infty}} &\lesssim \|\mathsf{P}_u X\|_{\mathcal{C}^{\frac{1}{2}}}\\ &\lesssim \|u\|_{\mathcal{C}^{-\frac{1}{4}}}\|X\|_{\mathcal{C}^{\frac{3}{4}}}\\ &\lesssim \|u^\sharp\|_{\mathcal{C}^{-\frac{1}{4}}}\|X\|_{\mathcal{C}^{\frac{3}{4}}}\\ &\lesssim \|u^\sharp\|_{B^\alpha_{p,\infty}}\|X\|_{\mathcal{C}^{\frac{3}{4}}}.\end{aligned}$$

This proves that $\|u\|_{B^\alpha_{p,\infty}} \lesssim \|u^\sharp\|_{B^\alpha_{p,\infty}}$, the other inequality being immediate. ▷

In Chapter IV, the above Lemma will be crucial as it allows to work at the level of $u^\sharp$ rather than u. From now on, we will denote by $\otimes$ the modified paraproduct, for which both the previous construction of $\mathcal{H}$ and Lemma I.6 with $\sigma = -\frac{1}{4}$ hold true. We also write $\circledcirc$ for the corresponding resonant operator and $\circledast = \otimes + \circledcirc$. Note that the choice $p > 16$ here is somehow arbitrary and comes from taking $\sigma = -\frac{1}{4}$ in our proof. We need however to fix a setting so that the choice of $\otimes$ do not depend on p later on, hence the threshold.
Back to the operator $\mathcal{H}$, the important idea is that the operator $\mathcal{H}\Gamma$ is a better behaved perturbation of the identity at the price that it is not self-adjoint anymore. Namely

$$\mathcal{H}\Gamma u^\# = -\Delta u^\# + G_\xi(u^\#)$$

where G_ξ is a bounded operator from[1] $W^{\sigma,p}$ to $W^{\sigma-1-\kappa,p}$ for any $\kappa > 0$ provided the resonant product $\Pi(u^\#, \xi)$ is well defined, i.e. $\sigma > 1+\kappa$.
We introduce the conjugaison of $\mathcal{H}$ via the map Γ, that is

$$\mathcal{H}^\sharp = \Gamma^{-1}\mathcal{H}\Gamma.$$

In particular, we have

$$\|(\mathcal{H}^\sharp + \Delta)u\|_{\mathcal{C}^{-\kappa}} \lesssim \|u\|_{\mathcal{C}^{1+\varepsilon}}$$

for any $\kappa, \varepsilon > 0$. This allows to compare the heat semigroup of associated to $\mathcal{H}^\sharp$ and Δ with

$$e^{-t\mathcal{H}^\sharp} - e^{t\Delta} = \int_0^t e^{(t-s)\Delta}(\mathcal{H}^\sharp + \Delta)e^{-s\mathcal{H}^\sharp} ds, \tag{I.5}$$

1. The cases $p = 2$ and $p = +\infty$ are proved in [MZ22] or [GIP15]. The case $p = 1$ follows from a straightforward modification of the computation in [MZ22], and the case of general p then follows from interpolation.

this is the main argument for the results in [BDM23]. The same argument for the Schrödinger equation with a second order paracontrolled expansion yields Strichartz inequalities, see [MZ22]. In particular, this perturbative argument gives Schauder estimates for the heat semigroup of $\mathcal{H}^\sharp$, that is

$$\|e^{-t\mathcal{H}^\sharp}u\|_{\mathcal{C}^\alpha} \lesssim t^{-\frac{\alpha-\beta}{2}}\|u\|_{\mathcal{C}^\beta}$$

for $\alpha \in (1,2)$ and $0 < \alpha-\beta < 2$, see [BDM23, Theorem 22]. This can be used to obtain Schauder estimates for the heat semigroup associated to $\mathcal{H}$. This will be important to prove local well-posedness for the Anderson stochastic quantization equation for deterministic initial data in Chapter III.

Proposition I.7 – *For any $\alpha,\beta \in (-1,1)$ such that $\alpha > \beta$, we have*

$$\|e^{-t\mathcal{H}}u\|_{\mathcal{C}^\alpha} \lesssim t^{-\frac{\alpha-\beta}{2}}\|u\|_{\mathcal{C}^\beta}. \tag{I.6}$$

Proof – First, one interpolates the bounds $\|e^{-t\mathcal{H}^\sharp}u\|_{\mathcal{C}^\beta} \lesssim \|u\|_{\mathcal{C}^\beta}$ and

$$\|e^{-t\mathcal{H}^\sharp}u\|_{\mathcal{C}^\alpha} \lesssim t^{-\frac{\alpha-\beta}{2}}\|u\|_{\mathcal{C}^\beta}$$

for $\alpha \in (1,2)$ and $0 < \alpha-\beta < 2$. Let $\theta \in (0,1)$ and consider

$$\gamma_\theta := \theta\alpha + (1-\theta)\beta \in (\beta,\alpha).$$

Interpolation gives

$$\begin{aligned}
\|e^{-t\mathcal{H}^\sharp}\|_{\mathcal{C}^\beta\to\mathcal{C}^{\gamma_\theta}} &\le \|e^{-t\mathcal{H}^\sharp}\|_{\mathcal{C}^\beta\to\mathcal{C}^\beta}^{1-\theta}\|e^{-t\mathcal{H}^\sharp}\|_{\mathcal{C}^\beta\to\mathcal{C}^\alpha}^{\theta} \\
&\lesssim t^{-\frac{\alpha-\beta}{2}\theta} \\
&\lesssim t^{-\frac{\gamma_\theta-\beta}{2}}
\end{aligned}$$

since $\gamma_\theta - \beta = \theta\alpha + (1-\theta)\beta - \beta = (\alpha-\beta)\theta$ hence

$$\|e^{-t\mathcal{H}^\sharp}u\|_{\mathcal{C}^\alpha} \lesssim t^{-\frac{\alpha-\beta}{2}}\|u\|_{\mathcal{C}^\beta}$$

for any $\alpha,\beta \in (-1,2)$ such that $\alpha > \beta$. For the result for $\mathcal{H}$, we have

$$\begin{aligned}
\|e^{-t\mathcal{H}}u\|_{\mathcal{C}^\alpha} &= \|\Gamma e^{-t\mathcal{H}^\sharp}\Gamma^{-1}u\|_{\mathcal{C}^\alpha} \\
&\lesssim \|e^{-t\mathcal{H}^\sharp}\Gamma^{-1}u\|_{\mathcal{C}^\alpha} \\
&\lesssim t^{-\frac{\alpha-\beta}{2}}\|\Gamma^{-1}u\|_{\mathcal{C}^\beta} \\
&\lesssim t^{-\frac{\alpha-\beta}{2}}\|u\|_{\mathcal{C}^\beta}
\end{aligned}$$

using that Γ and Γ^{-1} are continuous from $\mathcal{C}^\gamma$ to itself for $\gamma < 1$ and the result for $\mathcal{H}^\sharp$. ▷

Natural spaces are the general Sobolev spaces associated to $\mathcal{H}$ defined as the closure of the vector space spanned by the eigenfunctions of $\mathcal{H}$ with respect to the norm

$$\|u\|_{\mathcal{D}^\sigma} := \Big(\sum_{n\geq 1}\langle\lambda_n\rangle^\sigma|\langle u,\varphi_n\rangle|^2\Big)^{\frac{1}{2}} \tag{I.7}$$

for $\sigma\in\mathbb{R}$. In particular, one immediatly has the Schauder estimates

$$\|e^{-t\mathcal{H}}u\|_{\mathcal{D}^\alpha} \lesssim t^{-\frac{\alpha-\beta}{2}}\|u\|_{\mathcal{D}^\beta}$$

for $\alpha>\beta$. One can prove that $\mathcal{D}^\sigma = H^\sigma$ for $|\sigma|<1$, indeed, $\Phi : H^\sigma\to H^\sigma$ is invertible and $\mathcal{D}^\sigma = \Phi^{-1}H^\sigma$. Thus

$$\|e^{-t\mathcal{H}}u\|_{H^\alpha} \lesssim t^{-\frac{\alpha-\beta}{2}}\|u\|_{H^\beta}$$

for $\alpha,\beta\in(-1,1)$ such that $\alpha>\beta$. With the previous result, this gives the general Schauder estimates

$$\|e^{-t\mathcal{H}}u\|_{B^\alpha_{p,p}} \lesssim t^{-\frac{\alpha-\beta}{2}}\|u\|_{B^\beta_{p,p}}$$

for $p\in[2,\infty]$ and $\alpha,\beta\in(-1,1)$ such that $\alpha>\beta$ however we should only use the case $p=2$ or $p=\infty$ in this work.

Lemma I.8 – *For any $\sigma\in(-1,1)$ and $u\in\mathcal{C}^\sigma(\mathcal{M})$, the map $t\mapsto e^{-t\mathcal{H}}u\in\mathcal{C}^\sigma$ is continuous.*

Proof – First note that, by the semigroup property and (I.6), continuity at 0 is enough. Let $u\in\mathcal{C}^\sigma(\mathcal{M})$ and fix $\delta>0$. Then by density of $H^2(\mathcal{M})$ in $\mathcal{C}^\sigma(\mathcal{M})$ and continuity of Γ, we have that $\mathcal{D}(\mathcal{H})$ is dense in $\Gamma\mathcal{C}^\sigma(\mathcal{M}) = \mathcal{C}^\sigma(\mathcal{M})$. Thus there exists $v\in\mathcal{D}(\mathcal{H})$ such that $\|u-v\|_{\mathcal{C}^\sigma}<\delta$. Then we have for $h\geq 0$ small enough:

$$\Big\|(e^{-h\mathcal{H}}-1)u\Big\|_{\mathcal{C}^\sigma} \leq \Big\|(e^{-h\mathcal{H}}-1)(u-v)\Big\|_{\mathcal{C}^\sigma} + \Big\|(e^{-h\mathcal{H}}-1)v\Big\|_{\mathcal{C}^\sigma}.$$

As for the first term, using the Schauder estimate (I.6), we have

$$\Big\|(e^{-h\mathcal{H}}-1)(u-v)\Big\|_{\mathcal{C}^\sigma} \lesssim \|u-v\|_{\mathcal{C}^\sigma}\lesssim\delta,$$

uniformly in h. For the second term, we use

$$\begin{aligned}\Big\|(e^{-h\mathcal{H}}-1)v\Big\|_{\mathcal{C}^\sigma} &\lesssim \Big\|(e^{-h\mathcal{H}}-1)v\Big\|_{\mathcal{D}^{1+\sigma+\kappa}}\\ &\lesssim h^{\frac{1-\sigma-\kappa}{2}}\|v\|_{\mathcal{D}^2}.\end{aligned}$$

Taking h small enough depending on δ, this finally shows that

$$\Big\|(e^{-h\mathcal{H}}-1)u\Big\|_{\mathcal{C}^\sigma}\leq C\delta.$$

▷

I.2.2 Anderson Green's function and GFF

In order to get a precise bound on the truncated Green function of the Anderson operator, we first show that $G^{\mathcal{H}}$ and $G^{1-\Delta}$ have the same singularity. This is done by comparing both resolvents.

Lemma I.9 – *Let $4 \leq p < \infty$ and $0 < \delta < \frac{1}{2}$. Then $\mathcal{H}^{-1} - (1-\Delta)^{-1}$ is bounded from $W^{-\frac{2}{p}-\delta,p'}$ to $W^{\frac{2}{p}+\delta,p}$. In particular $(G^{\mathcal{H}} - G^{1-\Delta})(x,y) \in L_y^\infty W_x^{\frac{2}{p}+\delta,p}$, and $G^{\mathcal{H}}(x,y) \in L_x^\infty W_y^{\frac{2}{p}-\delta,p}$.*

Proof – $\mathcal{H}$ being self-adjoint and positive thus invertible, an application of the resolvent formula yields

$$\begin{aligned}\mathcal{H}^{-1} - (1-\Delta)^{-1} &= (\mathcal{H}\Gamma)^{-1} - (1-\Delta)^{-1} + \mathcal{H}^{-1} - (\mathcal{H}\Gamma)^{-1} \\ &= (\mathcal{H}\Gamma)^{-1}((1-\Delta) - \mathcal{H}\Gamma)(1-\Delta)^{-1} + B(\mathcal{H}\Gamma)^{-1}\end{aligned}$$

following yet again the same paradigm as before: $\mathcal{H}$ is a perturbation of $-\Delta$ through the map Γ. By definition, Γ is a perturbation of the identity, namely

$$\Gamma = 1 + \sum_{k\geq 1} (\cdot \otimes X)^k =: 1 + B. \tag{I.8}$$

Note that, as $X \in \mathcal{C}^{1-\kappa}$ for any $\kappa > 0$, we can write a nice continuity estimate for B. Indeed, for any $\nu \geq 0$ and $p > 1$,

$$\|u \otimes X\|_{W^{1-\nu-\kappa,p}} \leq c(\nu,\kappa,p)\|u\|_{W^{-\nu,p}}\|X\|_{\mathcal{C}^{1-\kappa}}$$

where $\|X\|_{\mathcal{C}^{1-\kappa}}$ can be made arbitrarily small up to a random truncation of X as in the definition of Γ. This proves that

$$B : W^{-\nu,p} \to W^{1-\nu-\kappa,p}$$

is a bounded operator for any $\kappa > 0$ and $\nu \geq 0$.
As for $(\mathcal{H}\Gamma)^{-1}$, if $u \in \mathcal{D}^{\nu-2}$ and $v = (\mathcal{H}\Gamma)^{-1}u$ is the solution to $\mathcal{H}\Gamma v = -\Delta v + G_\xi v = u$, we have for any $\nu \in (1;2]$ by definition of $\mathcal{D}^\nu(\mathcal{H}) = \Gamma^{-1}H^\nu$:

$$\|v\|_{H^\nu} = \left\|\Gamma\mathcal{H}^{-1}u\right\|_{H^\nu} = \left\|\mathcal{H}^{-1}u\right\|_{\mathcal{D}^\nu} = \|u\|_{\mathcal{D}^{\nu-2}} \sim \|u\|_{H^{\nu-2}}$$

where in the last step we used that Γ is invertible on H^σ for $\sigma \in (-1;1)$.
Combining all this with the resolvent formula above we obtain some $\kappa > 0$ small enough:

$$\begin{aligned}\left\|\mathcal{H}^{-1}u - (1-\Delta)^{-1}u\right\|_{W^{\frac{2}{p}+\delta,p}} &\lesssim \left\|(\mathcal{H}\Gamma)^{-1}(1-G_\xi)(1-\Delta)^{-1}u\right\|_{H^{1+\delta}} + \left\|B(\mathcal{H}\Gamma)^{-1}u\right\|_{W^{\frac{2}{p}+\delta,p}} \\ &\lesssim \left\|(1-G_\xi)(1-\Delta)^{-1}u\right\|_{H^{-1+\delta}} + \left\|(\mathcal{H}\Gamma)^{-1}u\right\|_{W^{\frac{2}{p}+\delta+\kappa-1,p}} \\ &\lesssim \left\|(1-G_\xi)(1-\Delta)^{-1}u\right\|_{W^{-\frac{2}{p}+\delta,p'}} + \left\|(\mathcal{H}\Gamma)^{-1}u\right\|_{H^{\delta+\kappa}}\end{aligned}$$

$$\begin{aligned}
&\lesssim \left\|(1-\Delta)^{-1}u\right\|_{W^{\max(-\frac{2}{p}+\delta,0)+1+2\kappa,p'}} + \|u\|_{H^{\delta+\kappa-2}} \\
&\lesssim \|u\|_{W^{\max(-\frac{2}{p}+\delta,0)-1+2\kappa,p'}} + \|u\|_{W^{\delta+\kappa-\frac{2}{p}-1,p'}} \\
&\lesssim \|u\|_{W^{-\frac{2}{p}-\delta,p'}},
\end{aligned}$$

where in the last step we used the restrictions on p and δ in the statement of Lemma I.9. This shows the boundedness properties of $\mathcal{H}^{-1}-(1-\Delta)^{-1}$, thus the regularity of $G^{\mathcal{H}}-G^{1-\Delta}$. The regularity of $G^{\mathcal{H}}$ then follows from that of $G^{1-\Delta}$ since $G^{\mathcal{H}}-G^{1-\Delta}$ is smoother. ▷

Next, we investigate the properties of the Green function and its smoothened counterparts. Let us recall the following fundamental lemma that can be found in [ORTW20, Lemma 2.8 , 2.12 and 2.13] :

Lemma I.10 – *The following comparison estimates hold:*
(i) *There exists* $C>0$ *such that for any* $N\in\mathbb{N}$ *and* $(x,y)\in\mathcal{M}\times\mathcal{M}\setminus\mathrm{diag}$ *it holds*

$$\left|(\mathbf{P}_N\otimes\mathbf{P}_N)G^{1-\Delta}(x,y)+\frac{1}{2\pi}\log\Big(\mathbf{d}(x,y)+N^{-1}\Big)\right|\le C. \tag{I.9}$$

(ii) *For all* $0<\delta\ll 1$ *there exists* $C>0$ *such that for any* $N_1\le N_2$ *and* $(x,y)\in\mathcal{M}\times\mathcal{M}\setminus\mathrm{diag}$ *it holds for any* $j\in\{1;2\}$*:*

$$\begin{aligned}
&\left|(\mathbf{P}_{N_j}\otimes\mathbf{P}_{N_j})G^{1-\Delta}(x,y)-(\mathbf{P}_{N_1}\otimes\mathbf{P}_{N_2})G^{1-\Delta}(x,y)\right| \\
&\qquad\le C\min\Big\{-\log\Big(\mathbf{d}(x,y)+N_1^{-1}\Big)\vee 1; N_1^{\delta-1}\mathbf{d}(x,y)^{-1}\Big\}.
\end{aligned}\tag{I.10}$$

Combining Lemmas I.9 and I.10, we get the following result.

Corollary I.11 – *The following comparison estimates hold:*
(i) *There exists* $C>0$ *such that for any* $N\in\mathbb{N}$ *and* $(x,y)\in\mathcal{M}\times\mathcal{M}\setminus\mathrm{diag}$ *it holds*

$$\left|(\mathbf{P}_N\otimes\mathbf{P}_N)G^{\mathcal{H}}(x,y)+\frac{1}{2\pi}\log\Big(\mathbf{d}(x,y)+N^{-1}\Big)\right|\le C. \tag{I.11}$$

(ii) *For all* $0<\delta\ll 1$ *there exists* $C>0$ *such that for any* $N_1\le N_2$ *and* $(x,y)\in\mathcal{M}\times\mathcal{M}\setminus\mathrm{diag}$ *it holds for any* $j\in\{1;2\}$*:*

$$\begin{aligned}
&\left|(\mathbf{P}_{N_j}\otimes\mathbf{P}_{N_j})G^{\mathcal{H}}(x,y)-(\mathbf{P}_{N_1}\otimes\mathbf{P}_{N_2})G^{\mathcal{H}}(x,y)\right| \\
&\qquad\le C\min\Big\{-\log\Big(\mathbf{d}(x,y)+N_1^{-1}\Big)\vee 1; N_1^{\delta-1}\mathbf{d}(x,y)^{-1}\Big\}.
\end{aligned}\tag{I.12}$$

Proof – We start with the proof of (I.11). Due to (I.9), we have

$$(\mathbf{P}_N\otimes\mathbf{P}_N)G^{\mathcal{H}}(x,y)=(\mathbf{P}_N\otimes\mathbf{P}_N)\Big(G^{\mathcal{H}}-G^{1-\Delta}\Big)(x,y)-\frac{1}{2\pi}\log\Big(\mathbf{d}(x,y)+N^{-1}\Big)+O_{L^\infty}(1).$$

Since $\mathbf{P}_N$ is uniformly bounded on $L^\infty(\mathcal{M})$, we also have

$$\Big\|(\mathbf{P}_N\otimes\mathbf{P}_N)\big(G^{\mathcal{H}}-G^{1-\Delta}\big)\Big\|_{L^\infty}\lesssim\|G^{\mathcal{H}}-G^{1-\Delta}\|_{L^\infty}\lesssim 1$$

uniformly in $N\geq 1$, due to Lemma I.9. This proves (I.11).
As for (I.12), we first show that

$$\Big\|\mathbf{P}_{N_1}-\mathbf{P}_{N_2}\Big\|_{W^{\sigma+\delta,p}\to W^{\sigma,p}}\lesssim N_1^{-\delta}\tag{I.13}$$

for any $N_1\leq N_2$, $\delta\in[0;2]$, $\sigma\in\mathbb{R}$, and $1\leq p\leq\infty$. Indeed, for any λ_n, we have from the mean value theorem

$$\psi(N_1^{-2}\lambda_n)-\psi(N_2^{-2}\lambda_n)=(N_1^{-2}-N_2^{-2})\lambda_n\int_0^1\psi'\Big(\big(\theta N_1^{-2}+(1-\theta)N_2^{-2}\big)\lambda_n\Big)d\theta,\tag{I.14}$$

Since ψ' is Schwartz, we get from (I.14) and (A.7) that

$$\begin{aligned}\Big\|\mathbf{P}_{N_1}-\mathbf{P}_{N_2}\Big\|_{W^{\sigma+2,p}\to W^{\sigma,p}}&\leq(N_1^{-2}-N_2^{-2})\int_0^1\Big\|\Delta\psi'\Big(\big(\theta N_1^{-2}+(1-\theta)N_2^{-2}\big)\Delta\Big)\Big\|_{W^{\sigma+2,p}\to W^{\sigma,p}}d\theta\\&\lesssim N_1^{-2}\int_0^1\Big\|\psi'\Big(\big(\theta N_1^{-2}+(1-\theta)N_2^{-2}\big)\Delta\Big)\Big\|_{W^{\sigma+2,p}\to W^{\sigma+2,p}}d\theta\\&\lesssim N_1^{-2},\end{aligned}\tag{I.15}$$

while using directly (A.7),

$$\Big\|\mathbf{P}_{N_1}-\mathbf{P}_{N_2}\Big\|_{W^{\sigma,p}\to W^{\sigma,p}}\leq\Big\|\mathbf{P}_{N_1}\Big\|_{W^{\sigma,p}\to W^{\sigma,p}}+\Big\|\mathbf{P}_{N_2}\Big\|_{W^{\sigma,p}\to W^{\sigma,p}}\lesssim 1.\tag{I.16}$$

Thus (I.13) follows from interpolating (I.15) and (I.16).
We can then write

$$\begin{aligned}&(\mathbf{P}_{N_1}\otimes\mathbf{P}_{N_1})G^{\mathcal{H}}(x,y)-(\mathbf{P}_{N_1}\otimes\mathbf{P}_{N_2})G^{\mathcal{H}}(x,y)\\&=(\mathbf{P}_{N_1}\otimes\mathbf{P}_{N_1})G^{1-\Delta}(x,y)-(\mathbf{P}_{N_1}\otimes\mathbf{P}_{N_2})G^{1-\Delta}(x,y)\\&\quad+\Big(\mathbf{P}_{N_1}\otimes(\mathbf{P}_{N_1}-\mathbf{P}_{N_2})\Big)(G^{\mathcal{H}}-G^{1-\Delta})(x,y).\end{aligned}$$

As for the terms on the first line, we estimate them with the use of (I.10), which is of course enough for (I.12). For the terms on the last line, since by Lemma I.9, $G^{\mathcal{H}}-G^{1-\Delta}\in L^\infty W^{\sigma,p}$ for $4\leq p<\infty$ and $\frac{2}{p}<\sigma<\frac{2}{p}+\frac{1}{2}$, we have together with Sobolev embedding and (I.13):

$$\Big\|\Big(\mathbf{P}_{N_2}\otimes(\mathbf{P}_{N_1}-\mathbf{P}_{N_2})\Big)(G^{\mathcal{H}}-G^{1-\Delta})\Big\|_{L^\infty_{x,y}}$$

$$\lesssim \Big\| \Big(\mathbf{P}_{N_2} \otimes (\mathbf{P}_{N_1} - \mathbf{P}_{N_2}) \Big) (G^{\mathcal{H}} - G^{1-\Delta}) \Big\|_{L^\infty_x W_y^{\frac{2}{p}+\varepsilon,p}}$$
$$\lesssim N_1^{-\delta} \Big\| G^{\mathcal{H}} - G^{1-\Delta} \Big\|_{L^\infty_x W_y^{\frac{2}{p}+\varepsilon+\delta,p}} \lesssim N_1^{-\delta}.$$

The difference $(\mathbf{P}_{N_2} \otimes \mathbf{P}_{N_2}) G^{\mathcal{H}} - (\mathbf{P}_{N_1} \otimes \mathbf{P}_{N_2}) G^{\mathcal{H}}$ is estimated similarly. This finally proves (I.12) provided that we take $\varepsilon, \delta > 0$ with $\varepsilon + \delta < \frac{1}{2}$. ▷

We have similar estimates if we use a "regularization" based on $\mathcal{H}$ instead of Δ.

Corollary I.12 – *Let $\psi \in \mathcal{S}(\mathbb{R})$ be as in (III.5).*
(i) *There exists $C > 0$ such that for any $N \in \mathbb{N}$ and $(x, y) \in \mathcal{M} \times \mathcal{M} \setminus \mathrm{diag}$ it holds*

$$\Big| (\psi(N^{-2}\mathcal{H}) \otimes \psi(N^{-2}\mathcal{H})) G^{\mathcal{H}}(x, y) + \frac{1}{2\pi} \log \Big(\mathbf{d}(x, y) + N^{-1} \Big) \Big| \leq C. \tag{I.17}$$

(ii) *For all $0 < \delta \ll 1$ there exists $c, C > 0$ such that for any $1 \leq M_1 \leq M_2$ and $N \in \mathbb{N}^*$, it holds for any $j \in \{1; 2\}$:*

$$\Big\| (\mathbf{P}_N \psi(M_j^{-2}\mathcal{H}) \otimes \mathbf{P}_N \psi(M_j^{-2}\mathcal{H})) G^{\mathcal{H}} - (\mathbf{P}_N \psi(M_1^{-2}\mathcal{H}) \otimes \mathbf{P}_N \psi(M_2^{-2}\mathcal{H})) G^{\mathcal{H}} \Big\|_{L^\infty_{x,y}}$$
$$\leq C N^{c\delta} M_1^{-\delta}. \tag{I.18}$$

Proof – For (I.17), we decompose

$$\begin{aligned}(\psi(N^{-2}\mathcal{H}) \otimes \psi(N^{-2}\mathcal{H})) G^{\mathcal{H}}(x, y) = (\mathbf{P}_N \otimes \mathbf{P}_N) G^{\mathcal{H}}(x, y) \\ + \Big(\mathbf{P}_N \otimes (\psi(N^{-2}\mathcal{H}) - \mathbf{P}_N) \Big) G^{\mathcal{H}}(x, y) \\ + \Big((\psi(N^{-2}\mathcal{H}) - \mathbf{P}_N) \otimes \psi(N^{-2}\mathcal{H}) \Big) G^{\mathcal{H}}(x, y).\end{aligned}$$

The first term is estimated by Corollary (I.11). As for the second term, we use the uniform boundedness of $\mathbf{P}_N$ on $L^\infty(\mathcal{M})$ and Lemma I.13 below to get for $0 < \varepsilon, \kappa \ll 1$

$$\Big\| \Big(\mathbf{P}_N \otimes (\psi(N^{-2}\mathcal{H}) - \mathbf{P}_N) \Big) G^{\mathcal{H}} \Big\|_{L^\infty_{x,y}} \lesssim \Big\| \Big(1 \otimes (\psi(N^{-2}\mathcal{H}) - \mathbf{P}_N) \Big) G^{\mathcal{H}} \Big\|_{L^\infty_x W_y^{\frac{2}{p}+\varepsilon,p}}$$
$$\lesssim \Big\| G^{\mathcal{H}} \Big\|_{L^\infty_x H_y^{\kappa+\varepsilon}} \lesssim 1.$$

Similarly, we have by symmetry and Lemma I.13 again that

$$\begin{aligned}&\Big\| \Big((\psi(N^{-2}\mathcal{H}) - \mathbf{P}_N) \otimes \psi(N^{-2}\mathcal{H}) \Big) G^{\mathcal{H}} \Big\|_{L^\infty_{x,y}} \\ &\quad\lesssim \Big\| \Big(1 \otimes (\psi(N^{-2}\mathcal{H}) - \mathbf{P}_N)^2 \Big) G^{\mathcal{H}} \Big\|_{L^\infty_x W_y^{\frac{2}{p}+\varepsilon,p}} + \Big\| \Big(\mathbf{P}_N \otimes (\psi(N^{-2}\mathcal{H}) - \mathbf{P}_N) \Big) G^{\mathcal{H}} \Big\|_{L^\infty_{x,y}} \\ &\quad\lesssim \Big\| \Big(1 \otimes (\psi(N^{-2}\mathcal{H}) - \mathbf{P}_N) \Big) G^{\mathcal{H}} \Big\|_{L^\infty_x H_y^{\kappa+\varepsilon}} + 1 \\ &\quad\lesssim 1.\end{aligned}$$

This shows (I.17).
As for (I.18), we start by showing the analogue of (I.13). From (I.14), we have

$$\begin{aligned}
&\Big\|\psi(M_1^{-2}\mathcal{H}) - \psi(M_2^{-2}\mathcal{H})\Big\|_{W^{\sigma+2+\varepsilon,p}\to W^{\sigma,p}} \\
&\qquad\le (M_1^{-2} - M_2^{-2})\int_0^1 \Big\|\mathcal{H}\psi'\Big((\theta M_1^{-1} + (1-\theta)M_2^{-2})\mathcal{H}\Big)\Big\|_{W^{\sigma+2+\varepsilon,p}\to W^{\sigma,p}} d\theta \\
&\qquad\lesssim (M_1^{-2} - M_2^{-2})\int_0^1 \Big\|\psi'\Big((\theta M_1^{-1} + (1-\theta)M_2^{-2})\mathcal{H}\Big)\Big\|_{W^{\sigma+2+\varepsilon,p}\to W^{\sigma+2,p}} d\theta \\
&\qquad\lesssim M_1^{-2},
\end{aligned}$$

thanks to Lemma I.13. Interpolating with

$$\Big\|\psi(M_1^{-2}\mathcal{H}) - \psi(M_2^{-2}\mathcal{H})\Big\|_{W^{\sigma+\varepsilon,p}\to W^{\sigma,p}} \lesssim 1$$

due to Lemma I.13 again, we get

$$\Big\|\psi(M_1^{-2}\mathcal{H}) - \psi(M_2^{-2}\mathcal{H})\Big\|_{W^{\sigma+\delta+\varepsilon,p}\to W^{\sigma,p}} \lesssim M_1^{-\delta}, \tag{I.19}$$

for any $\delta \in [0;2]$.
Then we estimate using Lemma A.7 and (I.19):

$$\begin{aligned}
&\Big\|(\mathbf{P}_N\psi(M_1^{-2}\mathcal{H}) \otimes \mathbf{P}_N\psi(M_1^{-2}\mathcal{H}))G^{\mathcal{H}} - (\mathbf{P}_N\psi(M_1^{-2}\mathcal{H}) \otimes \mathbf{P}_N\psi(M_2^{-2}\mathcal{H}))G^{\mathcal{H}}\Big\|_{L^\infty_{x,y}} \\
&\lesssim \Big\|(\mathbf{P}_N \otimes \mathbf{P}_N)\Big(\psi(M_1^{-2}\mathcal{H}) \otimes (\psi(M_1^{-2}\mathcal{H}) - \psi(M_2^{-2}\mathcal{H})\Big)G^{\mathcal{H}}\Big\|_{W_{x,y}^{\frac{2}{p}+\varepsilon,p}} \\
&\lesssim N^{-2(\frac{2}{p}+2\varepsilon)}\Big\|\Big(\psi(M_1^{-2}\mathcal{H}) \otimes (\psi(M_1^{-2}\mathcal{H}) - \psi(M_2^{-2}\mathcal{H})\Big)G^{\mathcal{H}}\Big\|_{W_{x,y}^{-\varepsilon,p}} \\
&\lesssim N^{2(\frac{2}{p}+2\varepsilon)}M_1^{-\frac{\varepsilon}{4}}\Big\|G^{\mathcal{H}}\Big\|_{W_x^{-\frac{\varepsilon}{2},p}W_y^{-\frac{\varepsilon}{4},p}} \\
&\lesssim N^{2(\frac{2}{p}+2\varepsilon)}M_1^{-\frac{\varepsilon}{4}}.
\end{aligned}$$

This shows (I.18) provided that we take p large enough and ε small enough. ▷

I.2.3 Comparison of semi-classical multipliers

In this part, we establish a comparison principle between Schwartz multipliers for the Anderson operator and Schwartz multipliers for the Laplace-Beltrami operator. This is similar to (A.15) in [BGT04], but there the authors deal with abstract symmetric perturbations of $-\Delta$ of order 1 in $\mathbb{R}^d$, whereas here we deal with a perturbation of order $1+\kappa$, $0<\kappa\ll 1$, with rough coefficients.

Lemma I.13 – *Let $\psi \in \mathcal{S}(\mathbb{R})$, $2 \le p < \infty$ and $\frac{2}{p}-1 < \sigma < \frac{2}{p}+1$, and $0<\kappa<\delta<1+\kappa$ and $\kappa-\delta \le \beta \le 2+\kappa-\delta$ such that $0 \le \sigma - \frac{2}{p} + \delta \le 2$. Then there exists $C>0$ such* *that for any $N \geq 1$ and $f \in H^\beta(\mathcal{M})$, it holds*

$$\left\|\psi(N^{-2}\mathcal{H})f - \psi(N^{-2}\Delta)f\right\|_{W^{\sigma,p}} \leq CN^{\max(\sigma+\kappa-\frac{2}{p}-\beta,0)}\|f\|_{H^\beta}. \tag{I.20}$$

Moreover, if ψ is compactly supported away from 0, it holds

$$\left\|\psi(N^{-2}\mathcal{H})f - \psi(N^{-2}\Delta)f\right\|_{W^{\sigma,p}} \leq CN^{\sigma+\kappa-\frac{2}{p}-\beta}\|f\|_{H^\beta}. \tag{I.21}$$

Proof – The proof closely follows that of [BGT04, Proposition 2.1 and Theorem 6]: by Helffer-Sjöstrand's formula, we have

$$\psi(N^{-2}\mathcal{H}) - \psi(-N^{-2}\Delta) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)\Big((z-N^{-2}\mathcal{H})^{-1} - (z+N^{-2}\Delta)^{-1}\Big)dz, \tag{I.22}$$

where $\widetilde{\psi}$ is an almost analytic extension of ψ. Here we take

$$\widetilde{\psi}(z) = \Big(\sum_{k=0}^{2}\frac{\psi^{(k)}(\operatorname{Re} z)}{k!}(i\operatorname{Im} z)^k\Big)\chi(\operatorname{Im} z)$$

with $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi \equiv 1$ near 0, so that

$$\overline{\partial}\widetilde{\psi}(z) = \frac{\widetilde{\psi}^{(2)}(\operatorname{Re} z)}{2}(i\operatorname{Im} z)^2\chi(\operatorname{Im} z) + \Big(\sum_{k=0}^{2}\frac{\widetilde{\psi}^{(k)}(\operatorname{Re} z)}{k!}(i\operatorname{Im} z)^k\Big)\chi'(\operatorname{Im} z),$$

providing

$$|\overline{\partial}\widetilde{\psi}(z)| \lesssim \langle \operatorname{Re} z\rangle^{-10}|\operatorname{Im} z|^2\mathbf{1}_{|\operatorname{Im} z|\lesssim 1}. \tag{I.23}$$

Then we decompose the right-hand side of (I.22) as

$$\begin{aligned}
-\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)\Big((z-N^{-2}\mathcal{H}^\sharp)^{-1} - (z+N^{-2}\Delta)^{-1}\Big)dz\Gamma^{-1}& \\
-\frac{1}{\pi}(\Gamma-1)\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)(z-N^{-2}\mathcal{H}^\sharp)^{-1}dz\Gamma^{-1}& \\
-\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)(z+N^{-2}\Delta)^{-1}dz(\Gamma^{-1}-1).&
\end{aligned} \tag{I.24}$$

We start by estimating the norm of the semi-classical resolvents in Sobolev spaces. First, for any $\nu \in (-1;2]$, since $\mathcal{D}^\nu = \Gamma H^\nu$, we have

$$\begin{aligned}
\left\|(z-N^{-2}\mathcal{H}^\sharp)^{-1}u\right\|_{H^\nu} \sim \left\|(z-N^{-2}\mathcal{H})^{-1}\Gamma u\right\|_{\mathcal{D}^\nu} &\sim \Big(\sum_{n\geq 0}\lambda_n^\nu\frac{|\langle\Gamma u,\varphi_n\rangle|^2}{|z+N^{-2}\lambda_n|^2}\Big)^2 \\
&\lesssim |\operatorname{Im} z|^{-1}\|\Gamma u\|_{D^\nu},
\end{aligned} \tag{I.25}$$

and similarly, for any $\nu \in \mathbb{R}$,

$$\Big\|(z+N^{-2}\Delta)^{-1}u\Big\|_{H^\nu} \lesssim |\operatorname{Im} z|^{-1}\|u\|_{H^\nu}, \tag{I.26}$$

uniformly in N. Moreover, if $u \in H^{\nu-2}(\mathcal{M})$, letting $v = (z+N^{-2}\Delta)^{-1}u$, we have $(1-\Delta)v = v + N^2(zv-u)$, from which we infer

$$\begin{aligned}\Big\|(z+N^{-2}\Delta)^{-1}u\Big\|_{H^\nu} &\lesssim (1+N^2|z|)\Big\|(z+N^{-2}\Delta)^{-1}u\Big\|_{H^{\nu-2}} + N^2\|u\|_{H^{\nu-2}}\\ &\lesssim |\operatorname{Im} z|^{-1}\langle z\rangle N^2\|u\|_{H^{\nu-2}}.\end{aligned} \tag{I.27}$$

Proceeding similarly, since $v = (z-N^{-2}\mathcal{H})^{-1}\Gamma u$ satisfies $\mathcal{H}v = zN^2v - N^2\Gamma u$, we have for any $\nu \in (-1;2]$:

$$\begin{aligned}\Big\|(z-N^{-2}\mathcal{H}^\sharp)^{-1}u\Big\|_{H^\nu} &\sim \Big\|(z-N^{-2}\mathcal{H})^{-1}\Gamma u\Big\|_{\mathcal{D}^\nu}\\ &\lesssim |z|N^2\Big\|(z-N^{-2}\mathcal{H})^{-1}\Gamma u\Big\|_{\mathcal{D}^{\nu-2}} + N^2\|\Gamma u\|_{\mathcal{D}^{\nu-2}}\\ &\lesssim |\operatorname{Im} z|^{-1}\langle z\rangle N^2\|\Gamma u\|_{\mathcal{D}^{\nu-2}},\end{aligned} \tag{I.28}$$

where we used that Γ is invertible on H^s, $s \in (-1;1)$ and $\Gamma H^s = \mathcal{D}^s$ for $s \in (-1;2]$. Interpolation between (I.25) and (I.28) on the one hand, and between (I.26) and (I.27) on the other hand, leads to both

$$\Big\|(z-N^{-2}\mathcal{H}^\sharp)^{-1}u\Big\|_{H^\nu} \lesssim |\operatorname{Im} z|^{-1}\langle z\rangle^{\frac{\eta}{2}}N^\eta\|\Gamma u\|_{\mathcal{D}^{\nu-\eta}} \tag{I.29}$$

for any $\nu \in (-1;2]$ and $\eta \in [0;2]$, and

$$\Big\|(z+N^{-2}\Delta)^{-1}u\Big\|_{H^\nu} \lesssim |\operatorname{Im} z|^{-1}\langle z\rangle^{\frac{\eta}{2}}N^\eta\|u\|_{H^{\nu-\eta}}, \tag{I.30}$$

for any $\nu \in \mathbb{R}$ and $\eta \in [0;2]$.
We can then estimate the terms in the right-hand side of (I.24). For the first term, we use the resolvent identity, (I.23), (I.29)-(I.30), and the mapping property of G_ξ and $\mathcal{H}\Gamma$, to get

$$\begin{aligned}&\Big\|\int_{\mathbb{C}}\bar\partial\widetilde{\psi}(z)\Big((z-N^{-2}\mathcal{H}^\sharp)^{-1}-(z+N^{-2}\Delta)^{-1}\Big)dz\Gamma^{-1}f\Big\|_{W^{\sigma,p}}\\ &\lesssim \int_{|\operatorname{Im} z|\lesssim 1}\langle z\rangle^{-10}|\operatorname{Im} z|^2\\ &\qquad\times\Big\|(z-N^{-2}\mathcal{H}^\sharp)^{-1}N^{-2}(G_\xi+(\mathcal{H}\Gamma\cdot)\otimes X)(z+N^{-2}\Delta)^{-1}\Gamma^{-1}f\Big\|_{H^{\sigma+1-\frac{2}{p}}}dz\\ &\lesssim N^{\sigma-\frac{2}{p}+\delta-2}\int_{|\operatorname{Im} z|\lesssim 1}\langle z\rangle^{\frac{\sigma}{2}-\frac{1}{p}+\frac{\delta}{2}-10}|\operatorname{Im} z|\Big\{\Big\|G_\xi(z+N^{-2}\Delta)^{-1}\Gamma^{-1}f\Big\|_{H^{1-\delta}}dz\\ &\qquad+\Big\|\Big(\mathcal{H}\Gamma(z+N^{-2}\Delta)^{-1}\Gamma^{-1}f\Big)\otimes X\Big\|_{H^{1-\delta}}\Big\}dz\end{aligned}$$

$$
\begin{aligned}
&\lesssim N^{\sigma-\frac{2}{p}+\delta-2}\int_{|\operatorname{Im} z|\lesssim 1}\langle z\rangle^{\frac{\sigma}{2}-\frac{1}{p}+\frac{\delta}{2}-10}|\operatorname{Im} z|\Big\{\Big\|(z+N^{-2}\Delta)^{-1}\Gamma^{-1}f\|_{H^{2-\delta+\kappa}}\\
&\qquad+\Big\|\mathcal{H}\Gamma(z+N^{-2}\Delta)^{-1}\Gamma^{-1}f\Big\|_{H^{\kappa-\delta}}\Big\}dz\\
&\lesssim N^{\sigma-\frac{2}{p}+\delta-2}\int_{|\operatorname{Im} z|\lesssim 1}\langle z\rangle^{\frac{\sigma}{2}-\frac{1}{p}+\frac{\delta}{2}-10}|\operatorname{Im} z|\Big\|(z+N^{-2}\Delta)^{-1}\Gamma^{-1}f\Big\|_{H^{2-\delta+\kappa}}dz\\
&\lesssim N^{\sigma-\frac{2}{p}+\kappa-\beta}\int_{|\operatorname{Im} z|\lesssim 1}\langle z\rangle^{\frac{\sigma+\kappa-\beta}{2}-\frac{1}{p}-10}\Big\|\Gamma^{-1}f\Big\|_{H^{\beta}}dz\\
&\lesssim N^{\sigma-\frac{2}{p}+\kappa-\beta}\|f\|_{H^{\beta}},
\end{aligned}
$$

where in the last step we used that Γ^{-1} is bounded on $H^{\beta}(\mathcal{M})$ by choice of β.
As for the second term in (I.24), we write again $\Gamma = 1+B$ as in (I.8), with B bounded from $W^{\sigma-1+\kappa,p}(\mathcal{M})$ to $W^{\sigma,p}(\mathcal{M})$. Using that $\psi\in\mathcal{S}(\mathbb{R})$, this term contributes as

$$
\begin{aligned}
\Big\|B\psi(N^{-2}\mathcal{H}^{\sharp})^{-1}\Gamma^{-1}f\Big\|_{W^{\sigma,p}} &\lesssim \Big\|\Gamma^{-1}\psi(N^{-2}\mathcal{H})f\Big\|_{W^{\sigma-1+\kappa,p}}\\
&\lesssim \Big\|\Gamma^{-1}\psi(N^{-2}\mathcal{H})f\Big\|_{H^{\sigma+\kappa-\frac{2}{p}}} \sim \Big\|\psi(N^{-2}\mathcal{H})f\Big\|_{\mathcal{D}^{\sigma+\kappa-\frac{2}{p}}}\\
&= \Big(\sum_{n\geq 0}\psi(N^{-2}\lambda_n)\lambda_n^{\sigma+\kappa-\frac{2}{p}}|\langle f,\varphi_n\rangle|^2\Big)^{\frac{1}{2}}\\
&\lesssim_L \Big(\sum_{n\geq 0}\langle N^{-2}\lambda_n\rangle^{-L}\lambda_n^{\sigma+\kappa-\frac{2}{p}}|\langle f,\varphi_n\rangle|^2\Big)^{\frac{1}{2}}\\
&\lesssim \Big(\sum_{\lambda_n\lesssim N^2}\lambda_n^{\sigma+\kappa-\frac{2}{p}}|\langle f,\varphi_n\rangle|^2\Big)^{\frac{1}{2}}\\
&\qquad + \Big(\sum_{\lambda_n\gg N^2}N^{2L}\lambda_n^{\sigma+\kappa-\frac{2}{p}-L}|\langle f,\varphi_n\rangle|^2\Big)^{\frac{1}{2}}\\
&\lesssim \Big(\sum_{\lambda_n\lesssim N^2}N^{2\max(\sigma+\kappa-\frac{2}{p}-\beta,0)}\lambda_n^{\beta}|\langle f,\varphi_n\rangle|^2\Big)^{\frac{1}{2}}\\
&\qquad + \Big(\sum_{\lambda_n\gg N^2}N^{2\max(\sigma+\kappa-\frac{2}{p}-\beta,0)}\lambda_n^{\beta}|\langle f,\varphi_n\rangle|^2\Big)^{\frac{1}{2}}\\
&\lesssim N^{\max(\sigma+\kappa-\frac{2}{p}-\beta,0)}\|f\|_{\mathcal{D}^{\beta}} \sim N^{\max(\sigma+\kappa-\frac{2}{p}-\beta,0)}\|f\|_{H^{\beta}}
\end{aligned}
$$

since $L\geq 0$ is arbitrary, and since $\mathcal{D}^{\beta}=H^{\beta}$ for the range of β in Lemma I.13.
At last, using (A.8), the third term in (I.24) is estimated similarly by

$$
\begin{aligned}
\Big\|\psi(N^{-2}\Delta)(f\varolessthan X)\Big\|_{W^{\sigma,p}} &\lesssim N^{\max(\sigma+\kappa-\frac{2}{p}-\beta,0)}\Big\|f\varolessthan X\Big\|_{B_{p,2}^{\beta+\frac{2}{p}-\kappa}}\\
&\lesssim N^{\max(\sigma+\kappa-\frac{2}{p}-\beta,0)}\|f\|_{B_{p,2}^{\beta+\frac{2}{p}-1}} \lesssim N^{\max(\sigma+\kappa-\frac{2}{p}-\beta,0)}\|f\|_{H^{\beta}}.
\end{aligned}
$$

Putting everything together finally proves (I.20).
As for (I.21), the only differences are that in treating the second term in the right-

hand side of (I.24), the sums on $\lambda_n \lesssim N^2$ and $\lambda_n \gg N^2$ are now reduced to a single sum on $\lambda_n \sim N^2$, for which the same bound holds with $N^{\max(\sigma+\kappa-\frac{2}{p}-\beta,0)}$ replaced by $N^{\sigma+\kappa-\frac{2}{p}-\beta}$. Similarly for the third term, using the definition of Besov spaces, that ψ is compactly supported away from zero, together with the definition (A.2) of the functional calculus, and (A.7):

$$\begin{aligned}\left\|\psi(N^{-2}\Delta)(f \circledcirc X)\right\|_{W^{\sigma,p}} &\lesssim \left\|\psi(N^{-2}\Delta)(f \circledcirc X)\right\|_{B^{\sigma}_{p,2}} \\ &= \left\|\psi(N^{-2}\Delta)M^{\sigma}\mathbf{1}_{M\sim N}\mathbf{Q}_M(f \circledcirc X)\right\|_{L^p_x\ell^2_M} \\ &\lesssim N^{\sigma+\kappa-\frac{2}{p}-\beta}\left\|M^{\beta+\frac{2}{p}-\kappa}\mathbf{1}_{M\sim N}\mathbf{Q}_M(f \circledcirc X)\right\|_{L^p_x\ell^2_M} \\ &\lesssim N^{\sigma+\kappa-\frac{2}{p}-\beta}\left\|(f \circledcirc X)\right\|_{B^{\beta+\frac{2}{p}-\kappa}_{p,2}} \\ &\lesssim N^{\sigma+\kappa-\frac{2}{p}-\beta}\left\|f\right\|_{B^{\beta+\frac{2}{p}-1}_{p,2}} \lesssim N^{\sigma+\kappa-\frac{2}{p}-\beta}\left\|f\right\|_{H^{\beta}}.\end{aligned}$$

This proves (I.21). ▷

As a Corollary, we will use the following uniform boundedness property of smooth projectors for $\mathcal{H}$ in Hölder spaces.

Corollary I.14 – *Let $\chi \in C_0^\infty(\mathbb{R})$. Then for any $\sigma \in (-1;1)$ and $0<\kappa \ll 1$ it holds*

$$\left\|\chi(M^{-2}\mathcal{H})\right\|_{\mathcal{C}^{\sigma+\kappa}\to\mathcal{C}^{\sigma}} \lesssim 1,$$

uniformly in $M \in \mathbb{N}^*$.

Proof – Applying Lemmas I.13 and (A.7), we get for $p \gg 1$:

$$\begin{aligned}\left\|\chi(M^{-2}\mathcal{H})\right\|_{\mathcal{C}^{\sigma+\kappa}\to\mathcal{C}^{\sigma}} &\leq \left\|\chi(M^{-2}\mathcal{H})-\chi(M^{-2}\Delta)\right\|_{\mathcal{C}^{\sigma+\kappa}\to\mathcal{C}^{\sigma}} + \left\|\chi(M^{-2}\Delta)\right\|_{\mathcal{C}^{\sigma+\kappa}\to\mathcal{C}^{\sigma}} \\ &\lesssim \left\|\chi(M^{-2}\mathcal{H})-\chi(M^{-2}\Delta)\right\|_{H^{\sigma+\kappa}\to W^{\sigma+\frac{2}{p}+\frac{\kappa}{2},p}} + M^{-\kappa} \\ &\lesssim 1+M^{-\kappa}.\end{aligned}$$

▷

II Variational methods for Singular elliptic PDEs

Contents

II.1 – Introduction

Let $(\mathcal{M}, g)$ stand for a closed (compact, connected, boundaryless) two dimensional Riemannian manifold. We study in this work two classes of singular stochastic elliptic equations with multiplicative spatial white noise and prove existence results for them in settings where one cannot use a fixed point formulation of the equations. We are even able to exhibit a class of equations that have infinitely many solutions. This comes in stark contrast with all the well-posedness results proved in the literature on such singular stochastic partial differential equations under a small parameter assumption. This typically takes the form of existence (and uniqueness) for small times in the case of parabolic equations, e.g. [Hai14, Corollary 9.3] or [GIP15, Theorem 5.4], and small noise or strict convexity of the nonlinearity, as in [OW19, Theorem 1.1] or [AVG20, Theorem 3] for elliptic equations. We prove our results using a setting where solutions are understood in a weak sense and by resorting to variants of the mountain pass theorem. The use of topological methods to get critical points of C^1 functionals provides a very efficient and robust approach. There are however interesting equations that cannot be written as the Euler-Lagrange equation of some functional. The use of Ghoussoub's notion of self-dual functional provides a setting to characterize solutions of a number of equations as minimizers of a large class of functionals. Tools from convex analysis are required to set the scene. Note that the recent work [ZD23] was the first to implement the variational method for the construction of solutions to elliptic equations associated with the Anderson operator, their approach relies on a direct method of calculus of variations and aims at proving some regularity results on said solutions.

Section II.2 recalls and proves all we need to know about the Anderson operator and its perturbations by L^p potentials. Section II.3 is dedicated to the study of the equation

$$\mathcal{H}u = au + f(\cdot, u), \tag{II.1}$$

with potentials $a \in L^p(\mathcal{M})$ for some $p > 1$, with Theorem II.13 and Theorem II.14 as our main results. The statement gives mild conditions under which Equation (II.1) has at least one weak solution. The second statement shows that an additional parity condition on the nonlinearity f entails the existence of infinitely many weak solutions. Section II.4 is dedicated to the study of the non-variational singular Choquard-Pekar equation on $\mathcal{M} = \mathbb{T}^2$

$$\mathcal{H}u = au + \Big(w \star |u|^p\Big)|u|^{q-2}u,$$

for $p \neq q$. We obtain an existence result in Theorem II.17 for some appropriate w, p and q.

II.2 – Perturbations of the Anderson operator

We recall in Subsection II.2.1 a number of results about the Anderson operator and prove in Subsection II.2.2 that the quadratic form associated with the Schrödinger Anderson operator $\mathcal{H} + a$ has a nice spectral theory.

II.2.1 Basic results

We will only need on the Anderson operator the following crucial facts that we gether here for the sake of completness. We refer to Chapter I, [Mou22] and [BDM23] for the full story.

- $\mathcal{H}$ can be defined as a positive closed symmetric unbounded operator on $L^2(\mathcal{M})$ with domain $\mathcal{D}^2$ and compact resolvent. As such it has a nice spectral theory and $H : \mathcal{D}^2 \to L^2(\mathcal{M})$ is almost surely invertible. (See e.g. Section 2 of [Mou22] or Section 3 of [BDM23].)
- The form domain $\mathcal{D}^1$ of $\mathcal{H}$ is included and dense in any Sobolev space $H^\beta(\mathcal{M})$, for $0 \leq \beta < 1$, with compact inclusions.
- The operator $e^{-t\mathcal{H}}$ has a positive kernel $p_t^{\mathcal{H}}(x, y)$ and there exists positive (random) constants a_1, a_2 such that one has

$$\frac{1}{a_1 t} \exp\left(-a_2 \frac{d(x,y)^2}{t}\right) \leq p_t^{\mathcal{H}}(x,y) \leq \frac{a_1}{t} \exp\left(-\frac{d(x,y)^2}{a_2 t}\right), \tag{II.2}$$

  uniformly in $x, y \in \mathcal{M}$ and $t \in (0, 1]$, where $d(x, y)$ stands for the geodesic distance on $\mathcal{M}$ associated with the metric g. (See Proposition 25 in Section 4.3 of [BDM23].)

– There exists a positive (random) constant ε such that

$$e^{-t\mathcal{H}}\mathbf{1} \leq e^{-t\varepsilon}, \tag{II.3}$$

for all $t > 0$

– The Green function $G^{\mathcal{H}}$ of $\mathcal{H}$ has the same singularity as the one of $-\Delta + 1$ one the diagonal and satisfies

$$(G^{\mathcal{H}} - G^{1-\Delta})(x, y) \in L_y^\infty W_x^{\frac{2}{p}+\delta,p} \tag{II.4}$$

for $4 \leq p < +\infty$ and $0 < \delta < \frac{1}{2}$ (see Lemma I.9).

We do not record in the heat kernel $p_t^{\mathcal{H}}$ or the Green function $G^{\mathcal{H}}$ the dependence of these functions on the constant c as the latter will be fixed throughout. It follows from the second item and the Sobolev embedding that we have a compact inclusion of $\mathcal{D}^1$ into $L^q(\mathcal{M})$, for all $1 < q < \frac{2}{1-\alpha}$. Any bounded sequence in $\mathcal{D}^1$ has thus a subsequence that converges weakly in $\mathcal{D}^1$ and strongly in $L^q(\mathcal{M})$, for a given $1 < q < \frac{2}{1-\alpha}$. (We will use that fact a few times.) Do not be mislead by the comparison of the Green function of H with the Green function of Δ in the fourth item. While we have the comparison bound (II.4) between the two functions, the integral operator on functions associated with $G^{\mathcal{H}}$ does not have the regularizing properties that the operator Δ^{-1} have: there is *no* elliptic regularity for the operator $\mathcal{H}^{-1}$. This fact is related to the singular character of the Anderson operator and the low regularity of white noise.

It is already possible from these facts to say something about the solvability of the semilinear stationary Schrödinger Anderson equation

$$\mathcal{H}u = au + f(\cdot, u) \tag{II.5}$$

when the right hand side is a priori in $L^2(\mathcal{M})$, using the (almost sure) invertibility of $\mathcal{H}$ and the compact embedding of its domain in $L^2(\mathcal{M})$.

Proposition II.1 – *Assume that $a \in L^\infty(\mathcal{M})$ and that one can associate to $f \in C^0(\mathcal{M} \times \mathbb{R})$ a function $h \in L^2(\mathcal{M})$ such that $|f(\cdot, z)| \leq h(\cdot)$, uniformly in $z \in \mathbb{R}$. Then equation (II.5) has a solution if $\|a\|_{L^\infty}$ is small enough.*

Proof – The continuity of the operator $\mathcal{H}^{-1} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ and the estimate

$$\Big\|au + f(\cdot, u)\Big\|_{L^2} \leq \|a\|_\infty \|u\|_{L^2} + \|h\|_{L^2}$$

tell us that a ball of $L^2(\mathcal{M})$ of large enough radius is sent by the map $u \mapsto H^{-1}\Big(au + f(\cdot, u)\Big)$ into itself. As $\mathcal{H}^{-1}$ actually takes values in the compact subset $\mathcal{D}^2$ of $L^2(\mathcal{M})$ the conclusion comes from Schauder fixed point theorem. ▷

Alternatively, for $a \in L^2(\mathcal{M})$ one can use the Cameron-Martin theorem to say that the operator has a law that is equivalent to the law of $\mathcal{H}$. (One could even use the much refined form of Cameron-Martin theorem proved by Kusuoka for random potentials a, under appropriate assumptions – see e.g. Theorem 3.5.4 in [ÜZ00].) So the almost sure existence of a solution to equation (II.5) is equivalent in that case to the almost sure existence of a solution to equation

$$\mathcal{H}u = f(\cdot, u).$$

One can use a Schauder fixed point strategy if f satisfies for instance an estimate of the form

$$\left\| f(\cdot, u) \right\|_{L^2} \lesssim 1 + o(\|u\|_{L^2})$$

when $\|u\|_{L^2}$ goes to $+\infty$. This is in particular the case when $|f(\cdot, z)| \lesssim 1+|z|^{\ell-1}$, for $\ell \leq 2$. While the compactness/(fixed point) method is elementary to set up it requires in one form or another a small size or integrability assumption on a. The topological methods used in Section II.3 will bypass that constraint and work without size conditions on a for the much larger class of L^p potentials, for any $p > 1$. As a preliminary step to the developments of Section II.3 we first study the Schrödinger Anderson operator

$$u \mapsto (\mathcal{H} + a)u$$

for itself and give conditions on the potential a in the next section for its associated quadratic form to have a nice spectral theory. These conditions are met for a large class of potentials, including $a \in L^p(\mathcal{M})$ when $p > 1$.

II.2.2 The Kato class and the Anderson operator

The aim of this section is to prove the following diagonalisation result for the Schrödinger Anderson operator $\mathcal{H} + a$.

Theorem II.2 – *Pick $a \in L^p(\mathcal{M})$ with $p > 1$. There exists an orthonormal basis $(e_i)_{i\geq 0}$ of $L^2(\mathcal{M})$ such that*

$$\mathcal{D}^1 = \overline{\bigoplus_{i\geq 0} \mathbb{R}e_i},$$

with the closure in $\mathcal{D}^1$, and one has for all $i \geq 0$

$$\Big\langle e_i, (\mathcal{H} + a)e_i \Big\rangle_{L^2} = \mu_i.$$

Recall that a potential $a : \mathcal{M} \to \mathbb{R}$ is said to be in the *Kato class* if

$$\lim_{r\to 0^+} \sup_{x\in\mathcal{M}} \int_{d(x,\cdot)<r} \Big|\ln d(x, y)\Big| \, |a(y)| \, dy = 0. \tag{II.6}$$

Note that as $\mathcal{M}$ is compact, potentials in $L^p(\mathcal{M})$ with $p > 1$ are in the Kato class and that Kato class potentials are integrable. Given the equivalence (II.4) for the Green function $G^{\mathcal{H}}$ of the Anderson operator $\mathcal{H}$ one can rewrite condition (II.6) under the form

$$\lim_{r\to 0^+} \sup_{x\in\mathcal{M}} \int_{d(x,y)<r} G^{\mathcal{H}}(x, y)|a(y)| \, dy = 0.$$

The proof of Theorem II.2 follows the proof of a similar result for perturbations of the Δ operator by potentials in the Kato class. (See for instance Section 3.3 of the book [LHB20] of Betz, Hiroshima & Lorinczi.) We rewrite in Proposition II.4 condition (II.6) as a condition on the operator $\mathcal{H}+\lambda$, when the constant λ goes to ∞, and deduce from it in Proposition II.5 that the quadratic form associated with a is $\mathcal{H}$-form bounded with arbitrarily small relative bound. We first state and prove these two propositions before proving Theorem II.2. An intermediate result is needed first.

Lemma II.3 – *A function $a \in L^1(\mathcal{M})$ is in the Kato class iff*

$$\sup_{x\in\mathcal{M}} \int_0^T \int_{\mathcal{M}} p_s^{\mathcal{H}}(x,y)|a(y)|\, dyds \underset{T\to 0^+}{\longrightarrow} 0. \tag{II.7}$$

Proof – We first note from the Gaussian bounds (II.2) that condition (II.7) is equivalent to the condition

$$\sup_{x\in\mathcal{M}} \int_0^T \int_{\mathcal{M}} s^{-1}e^{-d(x,y)^2/s}|a(y)|\, dyds \underset{T\to 0^+}{\longrightarrow} 0. \tag{II.8}$$

• Let a be a potential in the Kato class. For $0<T<1$, we split the integration over $\mathcal{M}$ in (II.8) into $\{d(x,\cdot)<T^{1/4}\}\cup\{d(x,\cdot)\geq T^{1/4}\}$. By Fubini-Tonelli's theorem, a change of variables, and integration by parts, one has

$$\begin{aligned}\int_0^T \int_{d(x,\cdot)<T^{1/4}} s^{-1}e^{-d(x,y)^2/s}|a(y)|\, dyds &= \int_{d(x,\cdot)<T^{1/4}}\int_{T^{-1}d(x,y)^2}^{+\infty} r^{-1}e^{-r}|a(y)|\, drdy \\ &\lesssim -\int_{d(x,\cdot)<T^{1/4}} \ln\left(\frac{d(x,y)^2}{T}\right)|a(y)|\, dy + \int_{d(x,\cdot)<T^{1/4}}\int_{T^{-1}d(x,y)^2}^{+\infty} (\ln r)\, e^{-r}|a(y)|\, drdy \\ &\lesssim \int_{d(x,\cdot)<T^{1/4}} \big|\ln d(x,y)\big|\, |a(y)|\, dy + \ln T + o_T(1),\end{aligned}$$

with a $o_T(1)$ that comes from the integrable character of a and a negative contribution of $\ln T$ that can be skipped in an upper bound.
As we also have

$$\begin{aligned}\int_0^T \int_{d(x,\cdot)\geq T^{1/4}} s^{-1}e^{-d(x,y)^2/s}|a|(y)\, dyds &= \int_{d(x,\cdot)\geq T^{1/4}}\int_{T^{-1}d(x,y)^2}^{+\infty} r^{-1}e^{-r}|a|(y)\, dydr \\ &\leq \int_{d(x,\cdot)>T^{1/4}}\int_{T^{-1/2}}^{+\infty} r^{-1}e^{-r}|a|(y)\, drdy = o_T(1),\end{aligned}$$

from the fact that $a\in L^1(\mathcal{M})$, we see that condition (II.8) follows from condition (II.6).

- Write $A \asymp B$ when we have both $A \lesssim B$ and $B \lesssim A$. We have the estimate

$$\begin{aligned}\int_0^T p_s^{\mathcal{H}}(x,y)ds \asymp \int_0^T s^{-1}e^{-d(x,y)^2/s}ds \asymp \int_{d(x,y)^2/T}^{+\infty} r^{-1}e^{-r}dr \\ \asymp -\ln\Big(d(x,y)^2/T\Big)\, e^{-d(x,y)^2/T} + \int_{d(x,y)^2/T}^{+\infty} (\ln r)\, e^{-r}\, dr,\end{aligned}$$

which holds for any $0 < T < 1$ and uniformly in $x, y \in \mathcal{M}$, and thus the upper bound

$$\Big(-\ln d(x,y)\Big)\mathbf{1}_{d(x,y)\leq T} \lesssim \int_0^T p_s^{\mathcal{H}}(x,y)ds + \mathbf{1}_{d(x,y)\leq T}.$$

Multiplying by $|a|$, integrating on $\mathcal{M}$ and using again Fubini-Tonelli's theorem, we see on this inequality that condition (II.6) follows from condition (II.8). ▷

Proposition II.4 – *A function $a \in L^1(\mathcal{M})$ is in the Kato class iff $\|(\mathcal{H}+\lambda)^{-1}|a|\|_\infty \underset{\lambda\to+\infty}{\longrightarrow}$ 0.*

Proof – While the operator $(\mathcal{H}+\lambda)^{-1}$ is first defined as an operator from $L^2(\mathcal{M})$ into $\mathcal{D}^2$, for good λ's, its spectral representation

$$(\mathcal{H}+\lambda)^{-1}u = \int_0^{+\infty}\int_{\mathcal{M}} e^{-\lambda t}p_t^{\mathcal{H}}(\cdot,x)u(x)\,dxdt$$

allows to extend it naturally to the set of non-negative valued functions u, with $(\mathcal{H}+\lambda)^{-1}u$ taking values in $[0,+\infty]$. (Recall the heat kernel of $\mathcal{H}$ is positive, so the above quantity is positive unless u is null.) Take $T > 0$ to be chosen later. Slicing the time integral and changing variables, we have

$$\Big((\mathcal{H}+\lambda)^{-1}|a|\Big)(z) = \sum_{n\geq 0} e^{-T\lambda n}\int_0^T e^{-\lambda s}\int_{\mathcal{M}} p_{nT}^{\mathcal{H}}(z,x)\Big(e^{-s\mathcal{H}}|a|\Big)(x)\,dxds.$$

Thus with ε as in (II.3), we see from the fact that $e^{-nT\mathcal{H}}\mathbf{1} \leq e^{-nT\varepsilon}$ and the spectral representation of $(-H_c+\lambda)^{-1}$ that one has the upper bound

$$\begin{aligned}\Big((\mathcal{H}+\lambda)^{-1}|a|\Big)(z) &\leq \frac{1}{1-e^{-(\lambda+\varepsilon)T}}\sup_{x\in\mathcal{M}}\int_0^T \Big(e^{-s\mathcal{H}}|a|\Big)(x)\,ds \\ &\lesssim \frac{e^{\lambda T}}{1-e^{-(\lambda+\varepsilon)T}}\Big\|(\mathcal{H}+\lambda)^{-1}|a|\Big\|_\infty.\end{aligned}$$

Taking $T = 1/(\lambda+\varepsilon)$ shows then that we have

$$\Big\|(\mathcal{H}+\lambda)^{-1}|a|\Big\|_\infty \asymp \sup_{x\in\mathcal{M}}\int_0^T \Big(e^{-s\mathcal{H}}|a|\Big)(x)\,ds.$$

As
$$\int_0^T e^{-s\mathcal{H}}|a|(x)ds = \int_0^T \int_{\mathcal{M}} p_s^{\mathcal{H}}(y,x)|a|(y)\,dyds$$
and $p_s^{\mathcal{H}}(\cdot,\cdot)$ is a symmetric function of its two space arguments the quantity $\Big\|(\mathcal{H}+\lambda)^{-1}|a|\Big\|_\infty$ is equivalent to the quantity $\int_0^T\int_{\mathcal{M}} p_s^{\mathcal{H}}(x,y)|a|(y)\,dyds$, so the conclusion follows from Lemma II.3. ▷

Proposition II.5 – *Let a be a potential in the Kato class. For any $\eta>0$ there exists a positive constant m_η such that one has*
$$\langle u,|a|u\rangle_{L^2} \leq \eta\|u\|^2_{\mathcal{D}^1} + m_\eta\|u\|^2_{L^2},$$
for all $u\in\mathcal{D}^1$.

Proof – We prove below that the operator $|a|^{1/2}(\mathcal{H}+\lambda)^{-1/2}$ is well defined as an operator from $L^2(\mathcal{M})$ into itself, with operator norm of order $\Big\|(\mathcal{H}+\lambda)^{-1}|a|\Big\|_\infty^{1/2}$. The inequality of the statement then follows from the identity
$$\begin{aligned}\langle u,|a|u\rangle_{L^2} &= \||a|^{1/2}u\|^2_{L^2} = \Big\||a|^{1/2}(\mathcal{H}+\lambda)^{-1/2}(\mathcal{H}+\lambda)^{1/2}u\Big\|^2_{L^2}\\ &\leq \Big\||a|^{1/2}(\mathcal{H}+\lambda)^{-1/2}\Big\|^2_{L^2\to L^2}\Big\|(\mathcal{H}+\lambda)^{1/2}u\Big\|^2_{L^2},\end{aligned}$$
valid for $u\in\mathcal{D}^1$, and Proposition II.4.
Now note first that
$$\Big\|(\mathcal{H}+\lambda)^{-1}|a|\Big\|_{L^\infty\to L^\infty} = \Big\|(\mathcal{H}+\lambda)^{-1}|a|\Big\|_\infty.$$
By duality $|a|(\mathcal{H}+\lambda)^{-1}$ defines a bounded operator from $L^1(\mathcal{M})$ into itself, with operator norm
$$\Big\||a|(\mathcal{H}+\lambda)^{-1}\Big\|_{L^1\to L^1} = \Big\|(\mathcal{H}+\lambda)^{-1}|a|\Big\|_{L^\infty\to L^\infty} = \Big\|(\mathcal{H}+\lambda)^{-1}|a|\Big\|_\infty.$$
Stein's interpolation theorem can thus be applied to the holomorphic family of operators
$$T(z) := |a|^z(\mathcal{H}+\lambda)^{-1}|a|^{1-z},$$
and shows that $T(1/2)$ is a bounded operator from $L^2(\mathcal{M})$ into itself with operator norm at most $\|(\mathcal{H}+\lambda)^{-1}|a|\|_\infty$. The conclusion follows then from the identity
$$\Big\||a|^{1/2}(\mathcal{H}+\lambda)^{-1/2}\Big\|^2_{L^2\to L^2} = \Big\||a|^{1/2}(\mathcal{H}+\lambda)^{-1}|a|^{1/2}\Big\|_{L^2\to L^2} = \|T(1/2)\|_{L^2\to L^2}.$$
▷

The statement of Theorem II.2 is then a direct consequence of classical results on perturbations of quadratic forms, as Proposition II.5 allows us to use Theorem X.17 and

Theorem XIII.68 of Reed & Simon's books [RS75] and [RS78], respectively. We order the family of the real-valued (random) eigenvalues of the quadratic form $\mathcal{H} + a$

$$\mu_0 \leq \mu_1 \leq \cdots \leq \mu_m \leq 0 < \mu_{m+1} \leq \cdots \tag{II.9}$$

and denote by μ_{m+1} the smallest positive eigenvalue – with the convention that $m = -1$ if $\mu_0 > 0$. We record here for later use the following elementary result. Set

$$\mathcal{D}^1_{>m} := \overline{\bigoplus_{i \geq m+1} \mathbb{R} e_i},$$

with closure in $\mathcal{D}^1$.

Lemma II.6 – *Let $a \in L^p(\mathcal{M})$ for some $p > 1$. Then the following quantity is positive*

$$\delta := \inf_{\substack{v \in \mathcal{D}^1_{>m} \\ \|v\|_{\mathcal{D}^1} = 1}} \left(\|v\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} a v^2 \right) > 0.$$

Proof – We use the fact that $\alpha < 1$ can be chosen arbitrarily close to 1 to pick it in such a way that $2p/(p-1) < 2/(1-\alpha)$. Recall that the space $\mathcal{D}^1$ is compactly embedded in $H^\beta(\mathcal{M})$ for any $0 \leq \beta < \alpha$. Take a minimizing sequence u_n in $\mathcal{D}^1_{>m}$ with $\|u_n\|_{\mathcal{D}^1} = 1$, such that $\|u_n\|_{\mathcal{D}^1} + \int_{\mathcal{M}} a u_n^2 = 1 + \int_{\mathcal{M}} a u_n^2 \to \delta$. Then, since the sequence u_n is bounded in $\mathcal{D}^1$ and takes values in the closed subspace $\mathcal{D}^1_{>m}$ it has a subsequence that converges weakly to an element u of $\mathcal{D}^1_{>m}$ and, together with Sobolev embedding, strongly to u in $L^{2p/(p-1)}(\mathcal{M})$. The integrals $\int_{\mathcal{M}} a u_n^2$ then converge to $\int_{\mathcal{M}} a u^2$, and

$$\begin{aligned}
\delta &= 1 + \int_{\mathcal{M}} a u^2 \\
&= \liminf_{n \to \infty} \|u_n\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} a u^2 \\
&\geq \|u\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} a u^2.
\end{aligned}$$

If $u = 0$ we have $\delta = 1$, otherwise since $u \in \mathcal{D}^1_{>m}$ we have

$$\delta \geq \int_{\mathcal{M}} \left(\left(u \mathcal{H} u + a u^2 \right) \geq \mu_{m+1} \|u\|^2_{\mathcal{D}^1}.\right.$$

▷

Remark 1 – Note that even if Theorem II.2 and Lemma II.6 would hold true for any Kato class potential a as well, we shall only consider the case where $a \in L^p(\mathcal{M})$ for some $p > 1$ in the following. This is required for our energy functional Φ defined below to be C^1.

II.3 – Weak solutions to singular stochastic PDEs

Let a function $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ be given, with $f(x,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R})$ for each $x \in \mathcal{M}$ and

$$\big|f(x,z)\big| \lesssim 1 + |z|^\ell,$$

for some positive exponent ℓ, uniformly in $x \in \mathcal{M}$. We associate to f the function

$$F(x,z) := \int_0^z f(x,r)\,dr, \tag{II.10}$$

defined for all for $(x,z) \in \mathcal{M} \times \mathbb{R}$. Pick $a \in L^p(\mathcal{M})$ with $p > 1$ and set

$$\Phi(u) := \frac{1}{2}\|u\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} \left(\frac{1}{2}\,a(x)u(x)^2 - F\Big(x, u(x)\Big)\right) dx.$$

Lemma II.7 – *The function Φ on $\mathcal{D}^1$ is well-defined and C^1, with Fréchet derivative*

$$\Phi'(u)(v) = \langle u, v\rangle_{\mathcal{D}^1} + \int_{\mathcal{M}} \Big(a(x)u(x)v(x) - f\Big(x, u(x)\Big)v(x)\Big) dx$$

Proof – We use again the fact that $\alpha < 1$ can be chosen arbitrarily close to 1 to pick it in such a way that $2p/(p-1) < 2/(1-\alpha)$ and $\ell + 1 < 2/(1-\alpha)$. The continuous embedding of $\mathcal{D}^1$ into $L^{2p/(p-1)}(\mathcal{M})$ then tells us that the integral $\int_{\mathcal{M}} au^2$ defines a C^1 function of $u \in \mathcal{D}^1$ with derivative $v \mapsto 2\int_{\mathcal{M}} auv$ at point u. Similar considerations give the Fréchet differentiability of $\int_{\mathcal{M}} F\Big(x, u(x)\Big)dx$ as a function of $u \in \mathcal{D}^1$ and the formula for its derivative. $\triangleright$

This result justifies the following definition.

Definition II.8 – *A* ***weak solution of the equation***

$$-H_c u + au = f(\cdot, u) \tag{II.11}$$

is a critical point of the map Φ.

Note that as we are working in a Hilbert space framework one can identify the Fréchet derivative of Φ at point u to its gradient in $\mathcal{D}^1$, still denoted by $\Phi'(u)$. In their recent work [ZD23] Duan & Zhang use a similar characterization of weak solutions to the same equation. However their approach only deals with constant potentials $a \equiv \mu$, for which the direct method in the calculus of variations applies since the energy functional is coercive. This also allows them to prove that their weak solution actually belong to $\mathcal{D}^2$ and even get Schauder estimates on said solution. However in our case the whole point is that even the existence of a minimizer is not straightforward, since the energy functional is no longer coercive when a is not constant and non-positive. We note also that

Igant, Otto, Ried & Tsatsoulis also used slightly earlier a variational characterization of solutions to a nonlocal singular equation in their work [IORT23]. Note that one cannot use any kind of bootstrap, or elliptic regularity result, to get that weak solutions of equation (II.11) are strong solutions of that equation, as this would require a to be an element of $L^2(\mathcal{M})$. Indeed, u cannot be expected to have more than $H^{1^-}(\mathcal{M})$ regularity as an element of the domain $\mathcal{D}^2$, thus for the product au to be in $L^2(\mathcal{M})$ the potential a would need to have enough integrability, which we do not assume. Here our argument covers any potential $a \in L^p(\mathcal{M})$ in the whole range $p > 1$.

II.3.1 The Mountain Pass strategy

We use a well-known variant of the mountain pass theorem to guarantee the existence of critical points of Φ under appropriate assumptions on f. We begin with an informal presentation of the Mountain pass theorems in the finite dimensional setting. Imagine two points located at the deepest points of two valleys surrounded by a ring of mountains (as in Figure II.3.1). Suppose you go from one point to the other, then you would search for an optimal route, that is, with the least amount of climbing. Such a route goes through a mountain pass, which is not a maximum of the surrounding landscape since you searched for the route of minimal rise. Nor is it a minimum because you just climbed up out of the valley to get to the mountain pass, hence it should be a saddle point of the local landscape.

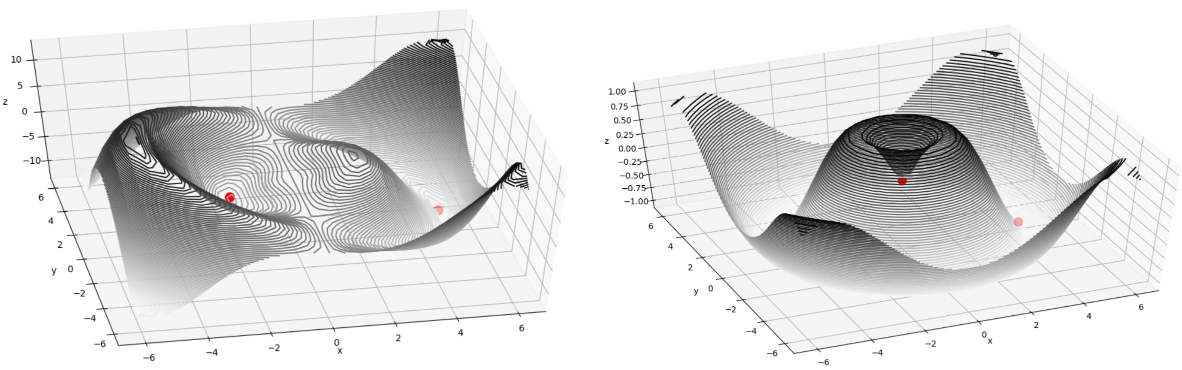

Figure II.3.1 – A Mountain pass landscape : the route with the least amount of climbing needs to go through a saddle point.

Proper assumptions are required on the landscape on which we wish to apply this idea, this is the prupose of this first recall the following definition.

Definition II.9 – *Let $b \in \mathbb{R}$. The functional Φ is said to satisfy the* ***Palais-Smale condition*** $(\textbf{\textit{PS}})_b$ *if any sequence (u_n) in $\mathcal{D}^1$ satisfying*

$$\Phi(u_n) \underset{n\to+\infty}{\longrightarrow} b, \qquad \Phi'(u_n) \underset{n\to+\infty}{\longrightarrow} 0, \tag{II.12}$$

has a converging subsequence in $\mathcal{D}^1$.

With this property, the general mechanics of minimax principles is simple and can be illustrated on the following special case.

Let B stand for the closed unit ball of the d-dimensional Euclidean space. Let ρ_0 be a continuous map from the unit sphere ∂B into $\mathcal{D}^1$. Let Γ stand for the collection of all continuous maps from B into $\mathcal{D}^1$ whose restriction to ∂B is ρ_0. If

$$\max_{|z|=1} \Phi\Big(\rho_0(z)\Big) < b := \inf_{\rho\in\Gamma} \|\Phi\circ\rho\|_\infty < \infty \tag{II.13}$$

then one can associate to every $\theta > 0$ and every $\rho \in \Gamma$ such that

$$\|\Phi\circ\rho\|_\infty \leq b + \theta$$

a point $u \in \mathcal{D}^1$ such that

$$\begin{cases} \Big|\Phi(u) - b\Big| \leq 2\theta, \\ \mathrm{dist}\Big(u, \rho(B)\Big) \leq 2, \\ \|\Phi'(u)\| \leq 8\theta. \end{cases}$$

Indeed if all points of the 2-neighbourhood of $\rho(B)$ where $\Big|\Phi(u) - b\Big| \leq 2\theta$ satisfied $\|\Phi'(u)\| > 8\theta$ one could build an explicit deformation $\widetilde{\rho}$ of ρ that would be in the family Γ and would satisfy $\|\Phi\circ\widetilde{\rho}\|_\infty \leq b-\theta$, contradicting the definition of b. Such a deformation would be constructed from the flow of a pseudo-gradient vector field associated with Φ'. See e.g. Lemma 2.2, Lemma 2.3 and Theorem 2.8 in Willem's book [Wil96] – here we took $\delta = 1$ in the notations of [Wil96]. So there exists a sequence of points $u_n \in \mathcal{D}^1$ satisfying

$$\Phi(u_n) \underset{n\to+\infty}{\longrightarrow} b, \qquad \Phi'(u_n) \underset{n\to+\infty}{\longrightarrow} 0.$$

If Φ satisfies the $(\mathsf{PS})_b$ condition, any limit point u is thus a critical point of Φ where $\Phi(u) = b$.

Let $S_r \subset \mathcal{D}^1$ stand for the sphere of $\mathcal{D}^1$ of radius r. If the maps $\rho \in \Gamma$ are of the form $\overline{\rho}\circ\iota$, where ι sends homeomorphically B into $\mathcal{D}^1$ and $\iota(B)\cap S_r \neq \emptyset$, with the maps $\overline{\rho}$ defined on $\iota(B)$, they satisfy $\rho(B)\cap S_r \neq \emptyset$ – otherwise one could construct a continuous retraction from B into ∂B. (See e.g. the proof of Theorem 2.12 in [Wil96].) Condition (II.13) thus holds true if

$$\max_{|z|=1} \Phi\Big(\rho(z)\Big) < \inf_{S_r} \Phi.$$

The (slightly refined) form under which we will use that fact is given by **Rabinowitz' linking theorem**, which we formulate in our setting here; see e.g. [Wil96, Theorem 2.12]. Set for all $k \geq 0$

$$\mathcal{D}^1_{\leq k} := \bigoplus_{i=0}^{k} \mathbb{R}e_i, \qquad \mathcal{D}^1_{>k} := \overline{\bigoplus_{i\geq k+1} \mathbb{R}e_i},$$

with closure in $\mathcal{D}^1$.

Theorem II.10 – *Pick $0 < r_1 < r_2 < \infty$ and $\mathsf{y} \in \mathcal{D}^1_{>k}$ with norm r_1. Set*

$$\mathcal{B}_{r_2} := \Big\{ u = y + t\mathsf{y}\,,\ y \in \mathcal{D}^1_{\leq k}\,;\, t \geq 0 \text{ such that } \|u\| \leq r_2 \Big\},$$

and let Γ stand for the set of continuous maps from $\mathcal{B}_{r_2}$ into $\mathcal{D}^1$ whose restriction to $\partial\mathcal{B}_{r_2}$ is the identity map. Then

$$b := \inf_{\rho\in\Gamma} \|\Phi \circ \rho\|_\infty$$

is a critical value of Φ if Φ satisfies the Palais-Smale condition $(\mathsf{PS})_b$ and

$$\max_{\partial\mathcal{B}_{r_2}} \Phi < \inf_{S_{r_1}\cap\mathcal{D}^1_{>k}} \Phi. \tag{II.14}$$

We will use that result to prove existence of weak solutions of equation (II.11). The following variation on Rabinowitz' linking theorem due to Bartsch will be used to prove that equation (II.11) actually have infinitely many solutions under an appropriate parity assumption on f. Here again the statement is given in our setting, and we refer e.g. to [Wil96, Theorem 3.6].

Theorem II.11 – *(Barstch's fountain Theorem)* *Assume f is odd with respect to its z argument. If Φ satisfies the Palais-Smale condition $(\mathsf{PS})_b$ for all $b \in \mathbb{R}$ and if there exist two sequences $0 < r_{1,n} < r_{2,n} < \infty$ such that*

$$\begin{cases} \displaystyle\max_{u\in\mathcal{D}^1_{\leq n}, |u|=r_{2,n}} \Phi(u) \leq 0, \\ \displaystyle\inf_{u\in\mathcal{D}^1_{>n}, |u|=r_{1,n}} \Phi(u) \underset{n\to+\infty}{\longrightarrow} +\infty, \end{cases}$$

then Φ has an unbounded sequence of critical values.

The parity condition on f implies that Φ is even, hence invariant by the action of the multiplicative group $\{\pm 1\}$. The role played by the (no retraction)/(Brouwer fixed point) argument in the proof of Theorem II.10 is played in that setting by the Borsuk-Ulam fixed point theorem. See e.g. Section 3.1 and Section 3.2 of [Wil96].

II.3.2 The Palais-Smale condition

We will work from now on with a nonlinearity $f \in C^1(\mathcal{M}\times\mathbb{R},\mathbb{R})$ that satisfies the following conditions, referred to in the text as **Assumption (A)**. Recall from (II.10) the definition of F.

- *There is an exponent $\ell > 2$ such that one has*

$$\big|f(x,z)\big| \lesssim 1 + |z|^{\ell-1}, \qquad \big|\partial_z f(x,z)\big| \lesssim 1 + |z|^{\ell-2},$$

and $f(x,z) = o(z)$, as z goes to 0, uniformly in $x \in \mathcal{M}$.

- *One has $F \geq 0$ and there exist $k > 0$ and $\gamma > 2$ such that for all $x \in \mathcal{M}$ one has*

$$\gamma F(x,z) \leq z f(x,z), \tag{II.15}$$

on the set $\{|z| \geq k\}$.

As an example, any focusing polynomial nonlinearity $f(x,z) = z^{2j+1}$ for an integer $j \geq 1$ satisfies Assumption (A).

Proposition II.12 – *The map Φ satisfies Palais-Smale condition $(\mathsf{PS})_b$ for all $b \in \mathbb{R}$.*

Proof – As a preliminary remark note that the differential condition (II.15) on the set $\{|z| > k\}$ gives the existence of positive constants c_1, c_2 such that one has the global lower bound

$$F(x,z) \geq c_1|z|^\gamma - c_2 \tag{II.16}$$

on all of $\mathcal{M} \times \mathbb{R}$. Recall from (II.9) the definition of the index m. Let now (u_n) be a sequence of elements of $\mathcal{D}^1$ such that $\sup_n \Phi(u_n) =: M < +\infty$ and $\Phi'(u_n)$ tends to 0. Write

$$u_n =: y_n + y_n' \in \mathcal{D}^1_{\leq m} \oplus \mathcal{D}^1_{>m}.$$

We will choose below a constant $\beta \in (\frac{1}{\gamma}, \frac{1}{2})$. Independently of this constant, one has for n large enough, say $n \geq n_0$, the inequality $|\Phi'(u_n)(v)| \leq \|v\|_{\mathcal{D}^1}$, for all $v \in \mathcal{D}^1$. One thus has for such indices

$$\begin{aligned} M + \|u_n\|_{\mathcal{D}^1} &\geq \Phi(u_n) - \beta\Phi'(u_n)(u_n) \\ &= \left(\frac{1}{2} - \beta\right)\left(\|u_n\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} a u_n^2\right) - \int_{\mathcal{M}} \Big(F(\cdot, u_n) - \beta f(\cdot, u_n)u_n\Big) \\ &\geq \left(\frac{1}{2} - \beta\right)\left(\|u_n\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} a u_n^2\right) + (\gamma\beta - 1)\Big(c_1\|u_n\|^\gamma_{L^\gamma} - c_2\Big), \end{aligned} \tag{II.17}$$

from Assumption (A) and (II.16). Since the decomposition $u_n = y_n + y_n'$ is orthogonal in $L^2(\mathcal{M})$ and the space $\mathcal{D}^1_{>m}$ is stable for the map $\mathcal{H} + a$ we can use the definition of μ_0 and δ in Lemma II.6, to get

$$\begin{aligned} \|u_n\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} a u_n^2 &= \|y_n\|^2_{\mathcal{D}^1} + \|y_n'\|^2_{\mathcal{D}^1} + 2\langle y_n, y_n'\rangle_{\mathcal{D}^1} + \int_{\mathcal{M}} a\Big(y_n^2 + (y_n')^2 + 2y_n y_n'\Big) \\ &= \left(\|y_n\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} a y_n^2\right) + \left(\|y_n'\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} a (y_n')^2\right) \\ &\quad + 2\underbrace{\left(\langle y_n, y_n'\rangle_{\mathcal{D}^1} + \int_{\mathcal{M}} a y_n y_n'\right)}_{=0} \\ &\geq \mu_0\|y_n\|^2_{L^2} + \delta\|y_n'\|^2_{\mathcal{D}^1}. \end{aligned}$$

For the L^γ norm in (II.17) we remark that since $\mathcal{M}$ is compact and $\gamma > 2$ the space

$L^\gamma(\mathcal{M})$ is a subspace of $L^2(\mathcal{M})$ with

$$\|u_n\|_{L^\gamma}^\gamma \gtrsim \|u_n\|_{L^2}^\gamma \gtrsim \Big(\|y_n\|_{L^2}^2 + \|y_n'\|_{L^2}^2\Big)^{\gamma/2} \gtrsim \|y_n\|_{L^2}^\gamma.$$

We thus have for n large enough the inequality

$$C + \|u_n\|_{\mathcal{D}^1} \geq \Big(\frac{1}{2} - \beta\Big)\Big(\mu_0\|y_n\|_{L^2}^2 + \delta\|y_n'\|_{\mathcal{D}^1}^2\Big) + c_1(\gamma\beta - 1)\|y_n\|_{L^2}^\gamma$$

for a positive constant C. Using the equivalence of the norms on the finite dimensional space $\mathcal{D}^1_{\leq m}$ where y_n lives and choosing $\beta < 1/2$ close enough to $1/2$ to have $\gamma\beta > 1$ and the constant in front of $\|y_n\|_{L^2}^\gamma$ in

$$C + \|u_n\|_{\mathcal{D}^1} \geq \Big(1/2 - \beta\Big)\delta\|y_n'\|_{\mathcal{D}^1}^2 + \Big(c_1(\gamma\beta - 1) + c_2\Big(1/2 - \beta\Big)\mu_0\Big)\|y_n\|_{L^2}^\gamma$$

positive, this implies that the sequence u_n is bounded in $\mathcal{D}^1$. Indeed, assume for instance $\|y_n\|_{\mathcal{D}^1}$ is not bounded, then, as $\gamma > 2$ the previous inequality rewrites

$$1 + \|y_n'\|_{\mathcal{D}^1} + \|y_n\|_{\mathcal{D}^1} \gtrsim \|y_n'\|_{\mathcal{D}^1}^2 + \|y_n\|_{\mathcal{D}^1}^2 \gtrsim \Big(\|y_n'\|_{\mathcal{D}^1} + \|y_n\|_{\mathcal{D}^1}\Big)^2$$

proving that $\|y_n'\|_{\mathcal{D}^1} + \|y_n\|_{\mathcal{D}^1}$ is bounded, which contradicts the fact that $\|y_n\|_{\mathcal{D}^1}$ is not bounded. Similar argument ensures that $\|y_n'\|_{\mathcal{D}^1}$ is bounded as well, and thus so is $\|u_n\|_{\mathcal{D}^1}$.

There is thus a subsequence $(u_{n'})$ that converges weakly to an element $u \in \mathcal{D}^1$ and in $L^{p/2}(\mathcal{M})$ and $L^{2p/(p-1)}(\mathcal{M})$ to u as well. We obtain the convergence of $u_{n'}$ to u in $\mathcal{D}^1$ from the identity

$$\Big(\Phi'(u_{n'}) - \Phi'(u)\Big)(u_{n'} - u) = \|u_{n'} - u\|_{\mathcal{D}^1}^2 - \int_{\mathcal{M}}\Big((f(\cdot, u_{n'}) - f(\cdot, u))(u_{n'} - u) - a(u_{n'} - u)^2\Big)$$

and the fact that

- the quantity $\Big(\Phi'(u_{n'}) - \Phi'(u)\Big)(u_{n'} - u)$ is converging to 0 since $u_{n'}$ is converging weakly to u in $\mathcal{D}^1$ and $\Phi'(u_{n'})$ is converging to 0,
- the two quantities $\int_{\mathcal{M}}\Big(f(\cdot, u_{n'}) - f(\cdot, u)\Big)(u_{n'} - u)$ and $\int_{\mathcal{M}} a(u_{n'} - u)^2$ are converging to 0 from Hölder inequality and the $L^{p/2}(\mathcal{M})$, respectively $L^{2p/(p-1)}(\mathcal{M})$, convergence of $u_{n'}$ to u.

This concludes the proof that Φ satisfies the Palais-Smale condition $(\mathsf{PS})_b$ for all $b \in \mathbb{R}$. ▷

II.3.3 Existence and multiplicity results

We can now state and prove our main existence and multiplicity results for the semilinear equation

$$\mathcal{H}u + au = f(\cdot, u). \tag{II.18}$$

Note that unlike in the fixed point approach of Proposition II.1 no small size or a good integrability assumption on a is needed in the next statement.

Theorem II.13 – *If f satisfies assumption **(A)**, then for any $a \in L^p(\mathcal{M})$ with $p > 1$, the equation (II.18) has a non-trivial weak solution in $\mathcal{D}^1$.*

Proof – Proposition II.12 shows that the map Φ satisfies the Palais-Smale condition $(\mathsf{PS})_b$ for all $b \in \mathbb{R}$. We now check the condition (II.14) of Rabinowitz' linking theorem, Theorem II.10, with $\mathsf{y} = r_1 \frac{e_{m+1}}{\|e_{m+1}\|_{\mathcal{D}^1}}$, for an appropriate choice of constants $0 < r_1 < r_2 < \infty$. We use the notations of Theorem II.10.
We have from the large and small z behaviour of $f(x,z)$ stated in Assumption **(A)** the existence for any $\theta > 0$ of a positive constant c_θ such that $|F(x,z)| \leq \theta |z|^2 + c_\theta |z|^\ell$, for all $(x,z) \in \mathcal{M} \times \mathbb{R}$. This gives in particular, for any $u \in \mathcal{D}^1_{>m}$, the lower bound

$$\begin{aligned} \Phi(u) &\geq \frac{\delta}{2} \|u\|^2_{\mathcal{D}^1} - \theta \|u\|^2_{L^2} - c_\theta \|u\|^\ell_{L^\ell} \\ &\geq \Big(\frac{\delta}{2} - \theta\Big) \|u\|^2_{\mathcal{D}^1} - c'_\theta \|u\|^\ell_{\mathcal{D}^1}, \end{aligned}$$

with δ as in Lemma II.6, and for another positive constant c'_θ, from the embedding of $\mathcal{D}^1$ in $L^\ell(\mathcal{M})$ when $\ell < 2/(1-\alpha)$. As $\ell > 2$ this inequality guarantees that for $0 < \theta < \delta/2$ and r_1 small enough

$$\inf_{S_{r_1} \cap \mathcal{D}^1_{>m}} \Phi > 0,$$

with S_{r_1} the sphere of $\mathcal{D}^1$ of radius r_1. We check in the sequel of the proof that one can find $r_2 > r_1$ finite such that

$$\sup_{\partial \mathcal{B}_{r_2}} \Phi \leq 0.$$

For $u \in \mathcal{D}^1_{\leq m}$ one has from the fact that F is non-negative and μ_m non-positive

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \Big(\|u\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} a u^2 \Big) - \int_{\mathcal{M}} F(\cdot, u) \\ &\leq \int_{\mathcal{M}} \Big(\frac{\mu_m}{2} u^2 - F(\cdot, u) \Big) \leq 0. \end{aligned}$$

For any $r_2 > 0$ and $u = y + t\mathsf{y} \in \mathcal{B}_{r_2}$ we have from the global lower bound (II.16) on F, and the equivalence of norms on the finite dimensional space $\mathcal{D}^1_{\leq m} \oplus \mathbb{R}\mathsf{y}$, the estimate

$$\begin{aligned} \Phi(u) &\leq \frac{1}{2} \|u\|^2_{\mathcal{D}^1} + \frac{1}{2} \|a\|_{L^p} \|u\|^2_{L^{2p/(p-1)}} - c_1 \int_{\mathcal{M}} |u|^\gamma + c_2 \\ &\lesssim \|u\|^2_{\mathcal{D}^1} + 1 - c'_1 \|u\|^\gamma_{\mathcal{D}^1}, \end{aligned}$$

for a positive constant c'_1. It follows that $\Phi(u) \leq 0$ if $\|u\|_{\mathcal{D}^1}$ is large enough, since $\gamma > 2$. The radius r_2 is chosen accordingly. $\triangleright$

The above proof males it clear that Theorem II.13 holds under the slightly weaker assumption that $F(\cdot, z)$ is only bounded below by $\mu_m z^2$.

Theorem II.14 – *Assume in addition to assumption* ***(A)*** *that f is odd with respect to its second argument. Then for any $a \in L^p(\mathcal{M})$ for some $p > 1$, there exists a sequence $(u_n) \subset \mathcal{D}^1$ of weak solutions of the equation*

$$\mathcal{H}u + au = f(\cdot, u)$$

such that $\Phi(u_n)$ goes to $+\infty$ as n goes to $+\infty$.

Proof – We check that the conditions of Bartsch's fountain theorem (Theorem II.11) are met. Given $n \geq m$ and $u \in \mathcal{D}^1_{>n}$, for $\theta < \frac{\delta}{2}$, as in the proof of Theorem II.13, we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\left(\|u\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} au^2\right) - \int_{\mathcal{M}} F(\cdot, u) \\ &\geq \frac{\delta}{2}\|u\|^2_{\mathcal{D}^1} - \theta\|u\|^2_{\mathcal{D}^1} - c_\theta\|u\|^\ell_{L^\ell} \geq \delta'\|u\|^2_{\mathcal{D}^1} - c_\theta\beta_n^\ell\|u\|^\ell_{\mathcal{D}^1}\end{aligned}$$

for a positive constant δ' and $\beta_n := \sup\limits_{u\in\mathcal{D}^1_{>n}} \frac{\|u\|_{L^\ell}}{\|u\|_{\mathcal{D}^1}}$. Set

$$r_{1,n}^{\ell-2} := \frac{\delta'}{2c_\theta\beta_n^\ell}$$

and take any $u \in \mathcal{D}^1_{>n}$ with $\|u\|_{\mathcal{D}^1} = r_{1,n}$. Then we have

$$\Phi(u) \geq r_{1,n}^2\left(\delta' - c_\theta\beta_n^\ell r_{1,n}^{\ell-2}\right) = \frac{\delta'}{2}r_{1,n}^2$$

In turns out that $r_{1,n}$ diverges to $+\infty$. To see this, note that the β_n are non-increasing so they have a limit $\beta \geq 0$. Pick for each $n \geq m$ a point $u_n \in \mathcal{D}^1_{>n}$ such that $\|u_n\|_{\mathcal{D}^1} = 1$ and $\|u_n\|_{L^\ell} \geq \beta_n/2$. Up to extraction, the u_n are converging weakly in $\mathcal{D}^1$ and in $L^\ell(\mathcal{M})$ to a limit element $u \in \mathcal{D}^1$. But it follows from the definition of $\mathcal{D}^1_{>n}$ that the u_n are converging weakly to 0, so $\beta = 0$ and $r_{1,n}$ diverges to $+\infty$.
To control the behaviour of Φ on $\mathcal{D}^1_{\leq n}$ we proceed as in the proof of Theorem II.13 and write for $u \in \mathcal{D}^1_{\leq n}$

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\left(\|u\|^2_{\mathcal{D}^1} + \int_{\mathcal{M}} au^2\right) - \int_{\mathcal{M}} F(\cdot, u) \\ &\leq \frac{1}{2}\|u\|^2_{\mathcal{D}^1} + \frac{1}{2}\|a\|_{L^p}\|u\|^2_{L^{2p/(p-1)}} - c_1\int_{\mathcal{M}} |u|^\gamma + c_2 \\ &\leq C_1(\|u\|^2_{\mathcal{D}^1} + 1) - C_{2,n}\|u\|^\gamma_{\mathcal{D}^1},\end{aligned}$$

for some positive constant C_1 and some n-dependent constant $C_{2,n}$, using the equivalence of norms on the finite dimensional space $\mathcal{D}^1_{\leq n}$. The condition $\gamma > 2$ thus guarantees that Φ takes non-positive values on the intersection with $\mathcal{D}^1_{\leq n}$ of the sphere of $\mathcal{D}^1$ of a well-chosen radius $r_{2,n} > r_{1,n}$. ▷

Corollary II.15 – *For any non-null even integer ℓ and any potential $a \in L^p(\mathcal{M})$, with $p > 1$, the semilinear problem*

$$\mathcal{H}u + au = u|u|^\ell$$

has infinitely many weak solutions.

Remark 2 – Due to $\mathcal{M}$ being compact and a being any $L^p(\mathcal{M})$ potential, the results in this section still hold true if we were not to include the smallest eigenvalue of $\mathcal{H}$ in its renormalization. We only need that the operator is semi-bounded from below.

II.4 – A non-variational singular stochastic PDE

We consider in this section the case of the two dimensional torus $\mathcal{M} = \mathbb{T}^2$. Denote by $\star$ the convolution operation in $\mathbb{T}^2$. We consider in this section the singular Choquard-Pekar equation

$$(\mathcal{H} + a)u = (w \star |u|^p)|u|^{q-2}u, \tag{II.19}$$

for appropriate parameters w, p, and q, which can be seen as a generalization of the (stationary) Hartree equation on $\mathbb{T}^2$ (for a survey on these equations in the deterministic case and the corresponding parameters, see [Ack04]). While the latter can be treated with variational methods, (II.19) cannot be written as the Euler-Lagrange equation of a functional on $\mathcal{D}^1$ as soon as $p \neq q$, the case of interest here. We use Ghoussoub's machinery of self-dual functionals to tackle that equation. We recall what we need from this setting in the restricted functional setting of the space $\mathcal{D}^1$ – this will be sufficient for us. See Ghoussoub's book [Gho08] for the whole story. It will clarify things here to make a difference between the Hilbert space $\mathcal{D}^1$ and its topological dual $\left(\mathcal{D}^1\right)' = \mathcal{D}^{-1}$ without identifying the later to the former. Do note that, contrary to the previous section, we will require the potential a to be positive. As such, it very much relies on the fact that we included positiveness of $\mathcal{H}$ in our renormalization procedure. The result would still hold true by shiffting the operator by its smallest (negative) eigenvalue.

Given a convex and lower semi continuous functional $\varphi : \mathcal{D}^1 \to \mathbb{R}$ its Fenchel transform $\varphi' : \mathcal{D}^{-1} \to \mathbb{R}$ is defined by

$$\varphi'(p) := \sup_{u \in \mathcal{D}^1} \Big(\langle p, u\rangle - \varphi(u)\Big),$$

and its subdifferential at a point $u \in \mathcal{D}^1$ is the subset of $\mathcal{D}^{-1}$ defined by

$$\partial\varphi(u) := \Big\{p \in \mathcal{D}^{-1}\,;\, \forall h \in \mathcal{D}^1, \varphi(u+h) \geq \varphi(u) + \langle p, u\rangle\Big\}.$$

One thus has

$$\varphi(u) + \varphi'(p) \geq \langle p, u\rangle$$

for all $u \in \mathcal{D}^1, p \in \mathcal{D}^{-1}$, with equality if and only if $p \in \partial\varphi(u)$. An operator $\Lambda : \mathcal{D}^1 \to \mathcal{D}^{-1}$ is said to be *regular* if it is weak-to-weak continuous on its domain and $u \mapsto (\Lambda u)(u)$ is weakly lower semi-continuous. Recall also that a non-negative function $\Phi : \mathcal{D}^1 \to [0, +\infty)$ is said to be *self-dual* if there exists a real-valued function M on $\mathcal{D}^1 \times \mathcal{D}^1$ such that

$$\Phi(u) = \sup M(u, \cdot) \tag{II.20}$$

for all $u \in \mathcal{D}^1$, where all the functions $M(v, \cdot)$ are proper and concave, all the functions $M(\cdot, v)$ are weakly semicontinuous and M is non-positive on the diagonal. A large class of self-dual functions is provided in the following statement. It is a direct consequence of Theorem 12.3 in Ghoussoub's book [Gho08] – itself a direct consequence of Ky Fan's min-max principle.

Theorem II.16 – *Let $\varphi : \mathcal{D}^1 \to \mathbb{R}$ be a lower semicontinuous convex function that is bounded below. Let $f \in \mathcal{D}^{-1}$ and $\Lambda : \mathcal{D}^1 \to \mathcal{D}^{-1}$ be a regular (possibly nonlinear) operator. Then the function*

$$M(u, v) := (\Lambda u)(u - v) + \varphi(u) - \varphi(v)$$

on $\mathcal{D}^1 \times \mathcal{D}^1$ defines via (II.20) a non-negative self-dual functional

$$\Phi(u) = \varphi(u) - \varphi'(\Lambda u) + (\Lambda u)(u).$$

If further $\left\|\varphi(u) + (\Lambda u)(u)\right\|_{\mathcal{D}^1}$ tends to $+\infty$ as $\|u\|_{\mathcal{D}^1}$ tends to $+\infty$ then the function Φ attains its minimum 0 at some point $\overline{u} \in \mathcal{D}^1$ where

$$-\Lambda\overline{u} \in \partial\varphi(\overline{u}).$$

We will use Theorem II.16 to prove the next statement, with φ the C^1 function

$$\varphi(u) := \frac{1}{2}\|u\|^2_{\mathcal{D}^1} + \frac{1}{2}\int_{\mathbb{T}^2} au^2$$

on $\mathcal{D}^1$. Its subdifferential being a singleton the conclusion of Theorem II.16 will thus come under the form that $\overline{u}$ is a weak solution of the equation

$$\partial\varphi(u) + \Lambda u = 0,$$

that is

$$(\mathcal{H} + a)\overline{u} + \Lambda\overline{u} = 0.$$

An ad hoc choice of function Λ will identify this equation with the Choquard-Pekar equation (II.19).

Theorem II.17 – *Pick exponents $p \in [1, +\infty)$, $q \in (1, +\infty)$ and let the potential a be* *bounded and positive. Assume that the interaction kernel $w \in L^1(\mathbb{T}^2)$ is non-positive. Then the singular Choquard-Pekar equation* (II.19) *has a weak solution $\overline{u} \in \mathcal{D}^1$.*

The non-positivity assumption on the interaction kernel w may seem rather ad hoc, as the Choquard-Pekar equation originally arose in the physics litterature for a non-*negative* kernel. However it also corresponds to the variational case $p = q$, and as we pointed out above, we rather see the equation (II.19) as a toy-model to extend the self-dual machinery to the case of a singular stochastic PDE. From this point of view our Theorem II.17 is really the analogue of [Gho08, Theorem 12.5 (A)].

Proof – The boundedness and positivity assumption on the potential a guarantees that the function φ on $\mathcal{D}^1$ is well-defined, convex, non-negative and lower semicontinuous. For $u \in \mathcal{D}^1$ set

$$\Lambda u := -(w \star |u|^p)|u|^{q-2}u.$$

One has for all $u, v \in \mathcal{D}^1$

$$\begin{aligned}
\Big|(\Lambda u)(v)\Big| &\leq \int_{\mathbb{T}^2} \Big|w \star |u|^p\Big|\, |u|^{q-1}|v| \\
&\leq \|w\|_{L^1}\, \||u|^p\|_{L^2}\, \Big\||u|^{q-1}v\Big\|_{L^2} \\
&\leq \|w\|_{L^1}\, \|u\|_{L^{2p}}^{p}\, \|u\|_{L^{2q}}^{q-1}\, \|v\|_{L^{2q}},
\end{aligned}$$

where we used Hölder inequality to bound $\Big\||u|^{q-1}v\Big\|_{L^2}$. Recall the compact embedding of $\mathcal{D}^1$ into $L^r(\mathbb{T}^2)$ for any $1 < r < \frac{2}{1-\alpha}$ mentioned in section 2.1 (where $\alpha - 2$ stands for the regularity of the white noise ξ). Using it for $r \in \{2p, 2q\}$ and an appropriate choice of α close enough to 1, then yields the bound

$$\Big|(\Lambda u)(v)\Big| \lesssim \|w\|_{L^1} \|u\|_{\mathcal{D}^1}^{p+q-1} \|v\|_{\mathcal{D}^1}$$

that shows that Λ is a well-defined map from $\mathcal{D}^1$ into $\mathcal{D}^{-1}$.
We check the weak-to-weak continuity of Λ. Let (u_n) converge weakly to u in $\mathcal{D}^1$. Let $v \in \mathcal{D}^1$. We have

$$\begin{aligned}
\Big|(\Lambda u_n)(v) - (\Lambda u)(v)\Big| &\leq \left|\int_{\mathbb{T}^2} \Big(w \star |u_n|^p\Big)|u_n|^{q-2}u_n v - \int_{\mathbb{T}^2} \Big(w \star |u|^p\Big)|u_n|^{q-2}u_n v\right| \\
&\quad + \left|\int_{\mathbb{T}^2} \Big(w \star |u|^p\Big)|u_n|^{q-2}u_n v - \int_{\mathbb{T}^2} \Big(w \star |u|^p\Big)|u|^{q-2}uv\right| \\
&\leq \left|\int_{\mathbb{T}^2} \Big(w \star \Big(|u_n|^p - |u|^p\Big)\Big)|u_n|^{q-2}u_n v\right| \\
&\quad + \left|\int_{\mathbb{T}^2} \Big(w \star |u|^p\Big)\Big(|u_n|^{q-2}u_n - |u|^{q-2}u\Big)v\right|
\end{aligned}$$

One can use once again the compact embedding of $\mathcal{D}^1$ into $L^r(\mathbb{T}^2)$ for some appropriate choice of $\alpha < 1$ and the weak convergence of u_n to u to get the convergence of $|u_n|^p$ to $|u|^p$ in $L^2(\mathbb{T}^2)$ and the convergence of $|u_n|^{q-2}u_n$ to $|u|^{q-2}u$ in $L^{2q/(q-1)}(\mathbb{T}^2)$. We therefore have

$$\begin{aligned}\Big|(\Lambda u_n)(v) - (\Lambda u)(v)\Big| \lesssim & \|v\|_{L^{2q}} \Big\| |u_n|^p - |u|^p \Big\|_{L^2} \|u_n\|_{L^{2q}}^{q-1} \\ & + \|v\|_{L^{2q}} \| |u|^p \|_{L^2} \Big\| |u_n|^{q-2} u_n - |u|^{q-2} u \Big\|_{L^{2q/(q-1)}}\end{aligned}$$

with an implicit multiplicative constant depending on $\|w\|_{L^1}$. The upper bound vanishes as n goes to ∞, which shows the weak-to-weak continuity. It follows in particular from the previous estimates that

$$\Big|(\Lambda u_n)(u_n) - (\Lambda u)(u)\Big| \leq \|w\|_{L^1} \Big\| |u_n|^p - |u|^p \Big\|_{L^2} \| |u_n|^q \|_{L^2} + \|w\|_{L^1} \| |u|^p \|_{L^2} \Big\| |u_n|^q - |u|^q \Big\|_{L^2}$$

goes to 0 as n goes to ∞. All this proves that the function Λ is regular. Remark at last that since the interaction kernel w is non-positive the function $\varphi(u) + (\Lambda u)u$ is coercive. We are thus in the setting of Theorem II.16, from which we get our conclusion. ▷

We note from the fact that Λ takes its values in $\mathcal{D}^{-1}$ that we could try and use a fixed point strategy to get a solution of equation (II.19), as in Proposition II.1 or the comment following it. In any case this would require that we assume either that a is small enough in $L^\infty(\mathbb{T}^2)$ or sufficiently integrable, and that w is small enough in $L^1(\mathbb{T}^2)$. The use of Theorem II.16 bypasses this kind of constraints. It is straightforward to adapt the proof of Theorem II.17 to the more general case of the equation

$$(\mathcal{H} + a)u = \big(w \star f(u)\big) g(u),$$

for nonlinearities $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ that are uniformly continuous and such that $\big|f(z)\big| \lesssim 1 + |z|^p$ and $\big|g(z)\big| \lesssim 1 + |z|^{q-1}$. So the existence result of Theorem II.17 holds in that setting.

III Anderson Stochastic Quantization Equation

Contents

III.1 – Introduction

III.1.1 A singular SPDE in a singular random environment

Let $\mathcal{M}$ be a closed Riemannian surface, 2-dimensional, compact and boundaryless. We consider the parabolic defocusing stochastic quantization equation

$$\begin{cases} \partial_t u + \mathcal{H}u + F'(u) = \sqrt{2}\zeta, \\ u(0) = u_0, \end{cases} \tag{III.1}$$

where

$$F(X) = \sum_{k=0}^{2m} a_k X^k \tag{III.2}$$

is a polynomial of even degree $2m$, ζ is a space-time white noise on $\mathbb{R} \times \mathcal{M}$, and $\mathcal{H}$ denotes the Anderson Hamiltonian operator. We assume that the noise ξ involved in the definition of $\mathcal{H}$ is independent of ζ and refer to Chapter I for the full story on the operator.

This study finds its place among the many other ones dealing with invariant measures for singular stochastic PDEs. If we were to get rid of the spatial white noise and consider

$-\Delta$ instead of $\mathcal{H}$, then (III.1) corresponds to the usual stochastic quantization equation of the $P(\Phi_2)$ model, for which the analogue of the Gibbs measure (III.3) was constructed as part of the development of the Euclidean Quantum Field Theory in the 60s and 70s (see e.g. [Sim74] and references therein), while the corresponding dynamics was first studied in [DD03]; see also [GH19; MW17a; TW18], among others. Other dynamics leaving this measure invariant were also studied: wave equations [GKO18; GKOT22; ORT23; OT20; OTWZ22], Schrödinger equations [BL23; Bou96; DNY19; OT18; Tzv08], and complex Ginzburg-Landau equations [DDF21; RZ; Tre19], to name but a few.

Without the stochastic forcing term ζ in (III.1) and with deterministic initial data, the equation becomes the Parabolic Anderson Model with polynomial nonlinearity, amenable to both the paracontrolled calculus approach [GIP15] or the approach using regularity structures [Hai14]. Recent progress on the understanding of the Anderson Hamiltonian [AC15; GIP15; GUZ20; Lab19; HL15; Mou22] opened the door to the study of dynamics in singular environment with regular enough deterministic data, such as the Anderson heat equation [GIP15; HL18], or Anderson wave and Schrödinger equations [DLTV23; DM19; DW18; GUZ20; MZ22; TV23b; TV23a; Ugu22]. In this paper, we go one step further and study a singular stochastic PDE in a singular random environment, namely the Anderson heat equation with polynomial nonlinearity and both rough deterministic or random initial data, and singular stochastic source term (III.1). A similar approach was recently considered by Barashkov, De Vecchi, and Zachhuber [BDZ23], who studied the problem of invariance of the Gibbs measure for the cubic non-linear Anderson wave equation.

III.1.2 Main results and sketch of the proof

Because ξ in (I.2) is almost surely of regularity $\mathcal{C}^{-1-\kappa}$ for any $\kappa > 0$, the very definition of $\mathcal{H}$ is tricky as one needs to go through a renormalization procedure to make sense of it as an unbounded $L^2(\mathcal{M})$ operator. Its precise definition is given in Section I.2, see for instance [AC15; GUZ20; Lab19; Mou22] and references therein for the whole story, the most important point being that $\mathcal{H}$ can be defined as an unbounded positive self-adjoint operator with dense domain and compact resolvent. In this work, we then construct a Gibbs type measure for (III.1) and prove local and almost sure global well-posedness and invariance of this measure. Note that all the probabilistic considerations we will consider are made conditionally to the randomness of $\mathcal{H}$.

The noiseless version of (III.1) has indeed a gradient flow structure

$$\partial_t u + \nabla_u \mathcal{E}(u) = 0,$$

with energy

$$\mathcal{E}(u) := \int_{\mathcal{M}} \left[\frac{1}{2} u\mathcal{H}u + F(u)\right] dx.$$

Following the finite dimensional mantra, this yields a formal Gibbs measure for (III.1) under the form

$$d\rho(u) = \mathcal{Z}^{-1} e^{-\mathcal{E}(u)} du,$$

or more precisely

$$d\rho(u) = \mathcal{Z}^{-1} e^{-\int_{\mathcal{M}} \left[\frac{1}{2}u\mathcal{H}u + F(u)\right] dx} du. \tag{III.3}$$

As it is, (III.3) is only a formal expression and several problems arise when we try to make sense of it. First of all, du supposedly refers to an infinite dimensional Lebesgue measure, which has no proper definition. The usual trick is to consider instead the quadratic part of the density together with du so that it defines a Gaussian measure

$$d\mu^{\mathcal{H}}(u) := \tilde{\mathcal{Z}}^{-1} e^{-\frac{1}{2}\int_{\mathcal{M}} u\mathcal{H}u\, dx} du,$$

see Subsection I.2.2 for the details on $\mu^{\mathcal{H}}$. It is defined as the centered Gaussian measure on $L^2(\mathcal{M})$ with covariance operator $\mathcal{H}^{-1}$. However, since $\mathcal{H}^{-1}$ is not trace class on $L^2(\mathcal{M})$, $\mu^{\mathcal{H}}$ will rather be supported on strictly larger spaces, namely $H^{-\varepsilon}(\mathcal{M})$ for any $\varepsilon > 0$. Another equivalent definition of $\mu^{\mathcal{H}}$ would be to match the law of the Gaussian Free Field (GFF) associated to $\mathcal{H}$, that is the random series

$$u^{\omega}(x) := \sum_{n \geq 0} \frac{\gamma_n(\omega)}{\sqrt{\lambda_n}} \varphi_n(x) \tag{III.4}$$

where γ_n are i.i.d. standard Gaussian random variables, λ_n the eigenvalues for $\mathcal{H}$ and φ_n the corresponding $L^2(\mathcal{M})$-orthonormal basis of eigenfunctions. This random series can be shown to almost surely converge in $H^{-\varepsilon}(\mathcal{M})$ for positive ε but not in $L^2(\mathcal{M})$. This poses another difficulty in that typical elements u in the support of $\mu^{\mathcal{H}}$ would then be distributions, making the non-linear term $F(u)$ ill-defined as $u \notin L^2(\mathcal{M})$ $\mu^{\mathcal{H}}$-almost surely. This can be handled by performing a Wick renormalization of the non-linear term $F(u)$ that will be described in Subsection III.2.2. In short, if

$$\mathbf{P}_N := \psi(N^{-2}\Delta) \tag{III.5}$$

is a Schwartz multiplier, then

$$\mathbb{E}_{\mu^{\mathcal{H}}}\left[|\mathbf{P}_N u|^2(x)\right] = \sum_{n \geq 0} \frac{1}{\lambda_n} |\mathbf{P}_N \varphi_n|^2(x) \to +\infty$$

as N goes to $+\infty$. This is due to the fact that the Green function of $\mathcal{H}$ on the diagonal has a logarithmic divergence, see (I.9) for a precise statement. Thus, Wick reordering replaces monomials X^k by Hermite polynomials $H_k(X, \sigma^2)$ with variance σ^2. Writing

$$\sigma_N^2(x) := \mathbb{E}_{\mu^{\mathcal{H}}}\left[|\mathbf{P}_N u|^2(x)\right], \tag{III.6}$$

we then define the corresponding renormalized monomial

$$(\mathbf{P}_N u)^{\diamond k} := H_k(\mathbf{P}_N u, \sigma_N^2)$$

and extend the definition to polynomials as in (III.2) by linearity:

$$F_N^\diamond(\mathbf{P}_N u) = \sum_{k=0}^{2m} a_k (\mathbf{P}_N u)^{\diamond k} = \sum_{k=0}^{2m} a_k H_k(\mathbf{P}_N u, \sigma_N^2). \tag{III.7}$$

The key point is that we get convergence of the renormalized non-linearity as $N \to +\infty$, that is

$$\int_{\mathcal{M}} F_N^\diamond(\mathbf{P}_N u)\, dx \to \int_{\mathcal{M}} F^\diamond(u)\, dx$$

in $L^p(d\mu^{\mathcal{H}})$ for any $p \geq 1$; see Proposition III.9 below. Definition and convergence properties of Hermite polynomials will be explained in Subsection III.2.2. We fix σ_N to be defined as in equation (III.6) and all the renormalized powers will be constructed with respect to σ_N. After this regularization and renormalization, we are left with an approximate Gibbs measure

$$d\rho_N(u) := \mathcal{Z}_N^{-1} e^{-\int_{\mathcal{M}} F_N^\diamond(\mathbf{P}_N u)\, dx}\, d\mu^{\mathcal{H}}(u). \tag{III.8}$$

Recall that we work with a defocusing non-linearity, thus ρ_N is well-defined for any fixed N. Our first result is the following, where we construct the Gibbs measure (III.3) as the limit of the sequence $(\rho_N)_N$.

Theorem III.1 – *The sequence $(\rho_N)_N$ converges in total variation to a measure ρ that is absolutely continuous with respect to $\mu^{\mathcal{H}}$.*

Detailed proof will be given in Subsection III.2.3 but the core idea, once convergence of the renormalized polynomials has been established, is to use Boué-Dupuis' formula as introduced in [BG20] and [BG23] to get uniform boundedness of the density.

Now that a candidate Gibbs measure has been constructed, we have to build the corresponding dynamics and check that the measure stays invariant under the flow. A natural way to do so is to truncate the equation

$$\partial_t u_N + \mathcal{H} u_N + \mathbf{P}_N f_N^\diamond(\mathbf{P}_N u_N) = \sqrt{2}\zeta \tag{III.9}$$

where $f_n^\diamond = (F_N^\diamond)'$. Local well-posedness can be proved using Da Prato Debussche trick, from there, it is only a matter of proving that solutions exist globally for some well chosen subset of initial data and that we can pass N to the limit. We first show local-well posedness and convergence of the solution to (III.9) for deterministic initial data.

Theorem III.2 – *Let $0 < \varepsilon < \frac{1}{2m-1}$, $\varepsilon < \sigma < 1$, and $q > 1$ be such that $\frac{\sigma+\varepsilon}{2}\frac{q}{q-1} < \frac{1}{2m-1}$. Then $\mathbb{P}$-almost surely, for any $u_0 \in \mathcal{C}^{-\varepsilon}(\mathcal{M})$ and $T_0 > 0$, there exists $T \in (0; T_0 \wedge 1]$ such that for any $N \in \mathbb{N}^*$, (III.9) admits a solution $u_N \in C([0;T]; \mathcal{C}^{-\varepsilon}(\mathcal{M}))$, unique in the affine space $\text{⫯} + X_T^{-\varepsilon,\sigma}$. Moreover u_N converges almost surely to some $u \in \text{⫯} + X_T^{-\varepsilon,\sigma}$ which is the unique solution in the class $\text{⫯} + X_T^{-\varepsilon,\sigma}$ to*[1]

$$\partial_t u + \mathcal{H} u + f^\diamond(u) = \sqrt{2}\zeta, \quad u(0) = u_0. \tag{III.10}$$

1. The renormalized nonlinearity $f^\diamond$ is well-defined on the class $\text{⫯} + X_T^{-\varepsilon,\sigma}$ as the limit of $\mathbf{P}_N f_N^\diamond(\mathbf{P}_N u)$ thanks to the algebraic rule in Lemma III.5 and the convergence of $(\mathbf{P}_N \text{⫯})^{\diamond k}$.

The space $X_T^{-\varepsilon,\sigma} \subset C([0;T];\mathcal{C}^{-\varepsilon}(\mathcal{M}))$ is defined in (III.26) below, and $\overset{\bullet}{|}$ is the stationnary solution to the linearized stochastic equation

$$(\partial_t + \mathcal{H})\overset{\bullet}{|} = \sqrt{2}\zeta.$$

Thus Theorem III.2 provides a solution for any data in $\mathcal{C}^{-\varepsilon}(\mathcal{M})$, which is the natural space where $\overset{\bullet}{|}(t)$ lives.

Our next result deals with globalization in the case where the initial data is random and distributed according to the Gibbs measure ρ constructed in Theorem III.1, and also proves that the law of the solution is invariant and given by ρ.

Theorem III.3 – *Let ε and σ be as in Theorem III.2. If u_0 is distributed according to ρ, then almost surely, for any $T > 0$ and $N \in \mathbb{N}^*$, the truncated equation (III.9) admits a unique solution in $\overset{\bullet}{|} + X_T^{-\varepsilon,\sigma}$. Moreover u_N converges in $C([0;T];\mathcal{C}^{-\varepsilon}(\mathcal{M}))$ to the unique solution $u \in \overset{\bullet}{|} + X_T^{-\varepsilon,\sigma}$ to (III.10), and the law of $u(t)$ does not depend on t and is given by ρ.*

As mentionned above, ζ denotes a space-time white noise on $\mathbb{R} \times \mathcal{M}$ that is independent of $\mathcal{H}$. In other words, ζ is a centered Gaussian process indexed by $L^2(\mathbb{R}\times\mathcal{M})$ functions and with covariance given by

$$\mathbb{E}\left[\zeta(\varphi)\zeta(\psi)\right] = \langle \varphi, \psi\rangle_{L^2(\mathbb{R}\times\mathcal{M})},$$

which is why ζ is said to have *delta* correlations since formaly the pointwise correlations are $\mathbb{E}\Big[\zeta(t,x)\zeta(s,y)\Big] = \delta_0(x-y)\delta_0(t-s)$. Note that given any $L^2(\mathcal{M})$ orthonormal basis $(\psi_n)_n$, we have the decomposition

$$\zeta(t)dt = \sum_{n\geq 0} dB_n(t)\psi_n$$

where $B_n(t) := \langle \zeta, \mathbb{1}_{[0,t]}\psi_n\rangle$ are i.i.d. standard Brownian motion on $\mathbb{R}$. Since $\mathcal{H}$ itself has an $L^2(\mathcal{M})$-orthonormal basis of eigenfunctions, the aforementioned expansion of ζ can be done using these eigenfunctions.

Remark 3 – As $\mathcal{H}$ is itself a random operator for which we fix a realisation $\mathcal{H}^{\omega'}$, the previous expansion of ζ reads

$$\zeta^\omega(t)dt = \sum_{n\geq 0} dB_n^{\omega,\omega'}(t)\varphi_n^{\omega'}$$

for a realisation ω. Note that ζ itself does not depend on ω' while its coefficients B_n do depend on both ω and ω'. In the following, we fix a realisation ω' so that $\mathcal{H}^{\omega'}$ is defined once and for all. In other words, all the probabilistic considerations have to be taken conditionally to $\mathcal{H}$. In particular, while we strongly believe that the double layer of randomness would be worth studying, it is out of the scope of this paper and we provide an approach that only relies on properties derived from the construction of $\mathcal{H}$.

The main difficulty in the solution theory of the renormalized equation (III.9) is then the operator $\mathcal{H}$ itself. On the one hand, the formal expression (I.2) can be made rigorous, namely $\mathcal{H}$ can be written (through conjugation with the so-called Γ map, see (I.4) in

Chapter I) as a perturbation of $-\Delta$. On the other hand, the perturbative terms have very rough coefficients, so that one has to carefully check that the classical analytic toolbox for the Laplace-Beltrami operator used in the context of more standard singular stochastic PDEs can be adapted to our setting despite those rough coefficients. In particular, we only dispose of a Schauder estimate with limited range of exponents (see (I.6) in Chapter I), and we have to use the precise mapping properties of the perturbative terms in $\mathcal{H}$ to show that the Green function for $\mathcal{H}$ has the same singularities as the one for $-\Delta$, which is crucial in building the Wick powers of the stochastic convolution to any degree; see Corollary I.11 and Proposition III.7.

We conclude this introduction with some further remarks.

Remark 4 – The solution constructed in Theorem III.2 for arbitrary initial data in $\mathcal{C}^{-\varepsilon}(\mathcal{M})$ is only local in time, while using the Gibbs measure we could globalize the solution only for ρ-almost every initial data in Theorem III.3. The question of deterministic global well-posedness and ergodicity of the dynamics is an interesting open question, and will be investigated in a follow-up work.

Remark 5 – We made sense of the formal equation (III.1) through a limiting procedure involving the equation with regularized nonlinearity (III.9). In the field of singular stochastic PDEs, it is more common to instead regularize the noise and replace ζ by some smooth approximation $\mathbf{P}_N\zeta$ and study the convergence of the corresponding solutions u_N to the renormalized equation

$$\partial_t u_N + \mathcal{H}u_N + f_N^{\diamond}(u_N) = \sqrt{2}\mathbf{P}_N\zeta$$

where now the renormalized nonlinearity $f_N^{\diamond}$ is not truncated by the operator $\mathbf{P}_N$ contrary to (III.9). However, (III.9) is better-suited for our purpose since (i) we want *in fine* to prove the invariance of the Gibbs measure, which we only know how to do by using a finite-dimensional approximation of the dynamics which has itself a finite-dimensional approximation of the Gibbs measure as invariant measure; and (ii) since $\mathcal{H}$ and $-\Delta$ do not commute, and since the regularizing operator $\mathbf{P}_N$ is a Schwartz multiplier based on $-\Delta$, it is not as straightforward to study the stochastic convolution $\widetilde{\mathbf{!}}_N(t) := \int_{-\infty}^t e^{-(t-s)\mathcal{H}}\mathbf{P}_N\zeta(ds)$ and its Wick powers to get the analogue of Proposition III.7.

Remark 6 – In (III.1), $\mathcal{H}$ is a Schrödinger operator with rough random electric potential ξ. The same arguments as those in the present paper can also deal with (III.1) where (I.2) is replaced by $(i\nabla + A)^2$, where now A is a rough and random magnetic potential. The construction of the corresponding magnetic Anderson operator when the magnetic field $B = \nabla \wedge A$ is a spatial white noise, namely $\nabla \wedge A = \xi$, has been carried out in [MM21]. In this case, (III.1) can be interpreted as the stochastic quantization equation for the marginal of the $P(\Phi_2)$-Yang-Mills-Higgs measure. Indeed, the abelian Yang-Mill-Higgs measure is the Gibbs type measure formally written as

$$\text{“}d\rho = \mathcal{Z}^{-\mathcal{E}(A,\Phi)}dAd\Phi\text{”},$$

where the abelian $P(\Phi_2)$-Yang-Mills-Higgs action is given by

$$\mathcal{E}(A,\Phi) = \frac{1}{2}\int_{\mathcal{M}} \Big[B^2 + |(i\nabla + A)\Phi|^2 + F(\Phi)\Big]dx;$$

see e.g. [She21]. Upon fixing a global gauge to have a one-to-one correspondence between A and B, we can then rewrite the measure similarly as in (III.3) by

$$\text{``}d\rho = \mathcal{Z}^{-1}e^{-\int_{\mathcal{M}}\left[\frac{1}{2}\Phi\mathcal{H}_B\Phi + F(\Phi)\right]dx}d\Phi d\mu(B),$$

where $\mathcal{H}_B = (i\nabla + A)^2$ and

$$d\mu(B) = \mathcal{Z}^{-1}e^{-\frac{1}{2}\int_{\mathcal{M}}B^2dx}dB$$

is the white noise measure. Given a realization of B, the measure $d\rho_B = \mathcal{Z}^{-1}e^{-\int_{\mathcal{M}}\left[\frac{1}{2}\Phi\mathcal{H}_B\Phi + F(\Phi)\right]dx}d\Phi$ is then the analogue of the measure (III.3), with the white noise electric potential ξ replaced by the white noise magnetic field B, and one can study its invariance under a parabolic singular stochastic dynamics similar to (III.1).

Remark 7 – Since for fixed time, the stochastic convolution for the Anderson stochastic heat equation and that of the Anderson stochastic damped wave equation

$$\partial_t^2 u + \mathcal{H}u + \partial_t u = \sqrt{2}\zeta \tag{III.11}$$

share the same covariance function (see e.g. [ORW21]), given by the Green function of $\mathcal{H}$, then Proposition III.7 also holds for the Wick powers of the solution to (III.11). Together with the same nonlinear analysis in L^2-based Sobolev spaces [2] as e.g. in [ORT23], this shows that the analogue of Theorem III.3 also holds true for the stochastic damped Anderson nonlinear wave equation with any polynomial nonlinearity

$$\partial_t^2 u + \mathcal{H}u + \partial_t u + f^{\diamond}(u) = \sqrt{2}\zeta.$$

Plan of the chapter

This chapter is organized as follows. In Section III.2 we construct the renormalized powers of the stochastic objects in play. We also use Boué-Dupuis formula to construct the Gibbs measure as the limit in total variation of its approximations (Theorem III.1). In Section III.3, we first prove local well-posedness for deterministic initial data in $\mathcal{C}^{-\varepsilon}(\mathcal{M})$, that is Theorem III.2. Then we prove invariance of the measure as well as global well-posedness for random initial data distributed according to the Gibbs measure, that is Theorem III.3.

2. We can simply use that $\mathcal{D}^{-\varepsilon} = H^{-\varepsilon}$ and $\mathcal{D}^{1-\varepsilon} = H^{1-\varepsilon}$, and that the propagator $\frac{\sin(t\sqrt{\mathcal{H}})}{\sqrt{\mathcal{H}}}$ maps $\mathcal{D}^{-\varepsilon}$ to $\mathcal{D}^{1-\varepsilon}$ from the functional calculus.

III.2 – Construction of the stochastic objects

In this section we provide the probabilistic framework to make sense of the renormalization procedure involved in the definition of ρ_N in (III.8) and also prove Theorem III.1.

III.2.1 Algebraic setting and Wick powers

We first of all recall the definition of Hermite polynomials and the corresponding algebraic rules, refering to [Nua06] or [DT23] for more insights on the whole story. We denote by H_k the k-th Hermite polynomial defined by the generating series

$$e^{tx-\frac{1}{2}t^2} = \sum_{k\geq 0} \frac{t^k}{k!} H_k(x)$$

for $x, t \in \mathbb{R}$. From this expansion, we easily get the recursion formula

$$H_k' = kH_{k-1}$$

for any positive integer k. Such polynomials are heavily related to Gaussian variables in the sense of the following lemma, see [Nua06, Lemma 1.1.1] for a proof.

Lemma III.4 – *Let X, Y be real standard Gaussian random variables. Then*

$$\mathbb{E}\left[H_k(X)H_l(Y)\right] = k!\delta_{k,l}\mathbb{E}[XY]^k.$$

Note that we have an extra $k!$ factor here, this is due to our normalization of Hermite polynomials so that they have leading coefficient 1 as opposed to $\frac{1}{k!}$ in [Nua06]. We also get the corresponding binomial formula.

Lemma III.5 – *For $x, y \in \mathbb{R}$ and $k \in \mathbb{N}^*$*

$$H_k(x+y) = \sum_{p=0}^{k} \binom{k}{p} x^{k-p} H_p(y).$$

Proof – It is a direct consequence of the definition of H_k by the generating series and the following expansion for $x, y, t \in \mathbb{R}$

$$\begin{aligned}
\sum_{k\geq 0} \frac{t^k}{k!} H_k(x+y) &= e^{(x+y)t-\frac{1}{2}t^2} = e^{xt}e^{yt-\frac{1}{2}t^2} \\
&= \sum_{n\geq 0} \frac{t^n}{n!} x^n \sum_{p\geq 0} \frac{t^p}{p!} H_p(y) \\
&= \sum_{k=n+p} \frac{t^k}{k!} \frac{k!}{p!n!} x^n H_p(y)
\end{aligned}$$

$$= \sum_{k\geq 0} \frac{t^k}{k!} \sum_{p=0}^{k} \binom{k}{p} x^{k-p} H_p(y).$$

▷

We then define the Hermite polynomials with variance $\sigma^2 > 0$ as

$$H_k(x, \sigma^2) := \sigma^k H_k(\sigma^{-1} x),$$

which allows to transfer the algebraic rules (III.4) and (III.5) to Gaussian random variables with non-unit variance. In the following, for a real centered Gaussian random variable X, we will use the shorthand notation called *Wick power* of X

$$X^{\diamond k} := H_k(X, \sigma^2),$$

where $\sigma^2 = \mathbb{E}[X^2]$. We end this subsection with the Wiener chaos estimate, see for example [Sim74, Theorem I.22].

Lemma III.6 – *Let $g = (g_n)$ be a sequence of i.i.d. real standard Gaussian random variables on some probability space Ω and (P_j) be a family of polynomials in g with degree at most $k \in \mathbb{N}$. Then for $p \geq 2$*

$$\Big\| \sum_{j\geq 0} P_j(g) \Big\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \Big\| \sum_{j\geq 0} P_j(g) \Big\|_{L^2(\Omega)}.$$

III.2.2 Renormalization

As progressing through the construction of the Gibbs measure (III.3), we will encounter truncated stochastic objects for which we would like to take the limit in the truncation parameter. This will either be for the non-linearity in the energy

$$\int_{\mathcal{M}} F_N^{\diamond}(\mathbf{P}_N u)\, dx,$$

where u is distributed according to the GFF $\mu^{\mathcal{H}}$ and $F_N^{\diamond}$ defined in (III.7), or for the stochastic convolution involved in the construction of the renormalized dynamic

$$\text{⫯}(t) := \sqrt{2} \int_{-\infty}^{t} e^{-(t-s)\mathcal{H}} \zeta(ds) \tag{III.12}$$

and its mollification

$$\text{⫯}_N(t) := \sqrt{2} \mathbf{P}_N \int_{-\infty}^{t} e^{-(t-s)\mathcal{H}} \zeta(ds).$$

Proposition III.7 – *Fix $k \in \mathbb{N}$, for any $T > 0$, $\varepsilon \in (0,1)$ and $p \geq q \geq 1$ the sequence $(\text{⫯}_N^{\diamond k})_N$ is a Cauchy sequence in $L^p(\Omega; L^q([0,T]; \mathcal{C}^{-\varepsilon}(\mathcal{M})))$ that also converges almost surely in $L^q([0,T]; \mathcal{C}^{-\varepsilon}(\mathcal{M}))$. Moreover $t \mapsto \text{⫯}_N(t)$ and the limiting process $t \mapsto \text{⫯}(t)$ are almost surely in $C([0,T]; \mathcal{C}^{-\varepsilon}(\mathcal{M}))$ and we have the following tail estimates and de-*

viation bounds:

$$\mathbb{P}\left(\|\Psi_N^{\diamond k}\|_{L^q_T\mathcal{C}^{-\varepsilon}_x} > R\right) \le Ce^{-cR^{\frac{2}{k}}T^{-\frac{2}{qk}}}, \tag{III.13}$$

and

$$\mathbb{P}\left(\|\Psi_N^{\diamond k} - \Psi^{\diamond k}\|_{L^q_T\mathcal{C}^{-\varepsilon}_x} > N^{-\kappa}R\right) \le Ce^{-cR^{\frac{2}{k}}T^{-\frac{2}{qk}}}, \tag{III.14}$$

for some constants $c, C > 0$, and exponent $\kappa > 0$ that do not depend on T, R, p, or N.

Proof – As a starter, we prove that $(\Psi_N^{\diamond k}(t))_N$ is bounded in $L^p(\Omega; \mathcal{C}^{-\varepsilon}(\mathcal{M}))$ uniformly in $N \in \mathbb{N}$ and $t \ge 0$. Using Itô's isometry and expanding ζ in the eigenbasis (φ_n) of $\mathcal{H}$, we get for $t \ge 0$, $x, y \in \mathcal{M}$ and $N_1, N_2 \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\left[\Psi_{N_1}(t,x)\Psi_{N_2}(t,y)\right] \\
&= 2\mathbb{E}\left[(\mathbf{P}^x_{N_1} \otimes \mathbf{P}^y_{N_2}) \int_{-\infty}^t \int_{-\infty}^t e^{-(t-s)\mathcal{H}}\zeta^x(ds)e^{-(t-\tau)\mathcal{H}}\zeta^y(d\tau)\right](x,y) \\
&= 2(\mathbf{P}^x_{N_1} \otimes \mathbf{P}^y_{N_2}) \sum_{n,m\ge 0} \mathbb{E}\left[\int_{-\infty}^t \int_{-\infty}^t e^{-(t-s)\lambda_n}\varphi_n^x dB_n(s) e^{-(t-\tau)\lambda_m}\varphi_m^y dB_m(\tau)\right](x,y) \\
&= 2(\mathbf{P}^x_{N_1} \otimes \mathbf{P}^y_{N_2}) \left(\sum_{n\ge 0}\int_{-\infty}^t e^{-2(t-s)\lambda_n}\varphi_n^x\varphi_n^y\, ds\right)(x,y) \\
&= \sum_{n\ge 0}\frac{(\mathbf{P}_{N_1}\varphi_n)(x)(\mathbf{P}_{N_2}\varphi_n)(y)}{\lambda_n} \\
&= G^{\mathcal{H}}_{N_1,N_2}(x,y),
\end{aligned}$$

where $G^{\mathcal{H}}_{N_1,N_2}(x,y) = (\mathbf{P}^x_{N_1} \otimes \mathbf{P}^y_{N_2})G^{\mathcal{H}}(x,y)$ and $G^{\mathcal{H}}$ is the Green function of $\mathcal{H}$. Since the covariance above does not depend on time, we drop the t dependency as it does not matter in the estimates. Combining the Sobolev inequality (Lemma A.3 (i) and (ii)) with the Wiener chaos estimate (III.6), we get for $\sigma \in (0,1)$ small enough depending on ε and p:

$$\begin{aligned}
\mathbb{E}\left[\|\Psi_N^{\diamond k}\|^p_{\mathcal{C}^{-\varepsilon}}\right] &\lesssim \mathbb{E}\left[\|\Psi_N^{\diamond k}\|^p_{W^{-\sigma,p}}\right] \\
&\lesssim \int_{\mathcal{M}} \mathbb{E}\left[\left((1-\Delta)^{-\frac{\sigma}{2}}(\Psi_N^{\diamond k})(x)\right)^p\right] dx \\
&\lesssim (p-1)^{p\frac{k}{2}} \int_{\mathcal{M}} \mathbb{E}\left[\left((1-\Delta)^{-\frac{\sigma}{2}}(\Psi_N^{\diamond k})(x)\right)^2\right]^{\frac{p}{2}} dx \\
&= (p-1)^{p\frac{k}{2}} \int_{\mathcal{M}} \left(\int_{\mathcal{M}^2} G_\sigma(x,y)G_\sigma(x,z)\mathbb{E}\left[\Psi_N^{\diamond k}(y)\Psi_N^{\diamond k}(z)\right] dy\, dz\right)^{\frac{p}{2}} dx,
\end{aligned}$$

where G_σ is the Green function of $(1-\Delta)^{-\frac{\sigma}{2}}$. From there, we use the previous computation and the algebraic rule (III.4) to get rid of the expectation so that

$$\mathbb{E}\left[\|{\stackrel{\bullet}{|}}_N^{\diamond k}\|_{\mathcal{C}^{-\varepsilon}}^p\right] \lesssim (p-1)^{p\frac{k}{2}} \int_{\mathcal{M}} \left(\int_{\mathcal{M}^2} G_\sigma(x,y) G_\sigma(x,z) G_N^{\mathcal{H}}(y,z)^k \, dy \, dz \right)^{\frac{p}{2}} dx.$$

Recall from (A.12) and (I.11) that

$$G_\sigma(x,y) \lesssim \mathbf{d}(x,y)^{\sigma-2}$$

and

$$\left| G_N^{\mathcal{H}}(x,y) + \frac{1}{2\pi} \log\left(\mathbf{d}(x,y) + N^{-1}\right) \right| \leq C.$$

Thus once we replace inside the integral it yields

$$\begin{aligned} &\mathbb{E}\left[\|{\stackrel{\bullet}{|}}_N^{\diamond k}\|_{\mathcal{C}^{-\varepsilon}}^p\right] \\ &\lesssim (p-1)^{p\frac{k}{2}} \int_{\mathcal{M}} \left(\int_{\mathcal{M}^2} \mathbf{d}(x,y)^{\sigma-2} \mathbf{d}(x,z)^{\sigma-2} (1 + |\log(\mathbf{d}(y,z) + N^{-1})|^k) \, dy \, dz \right)^{\frac{p}{2}} dx. \end{aligned}$$

Note that, since we integrate symmetric functions of $(y,z) \in \mathcal{M}^2$, we can restrict the integral to $\mathbf{d}(x,y) \geq \mathbf{d}(x,z)$ so that, using the triangle inequality on $\mathbf{d}(y,z)$,

$$\begin{aligned} \mathbb{E}\left[\|{\stackrel{\bullet}{|}}_N^{\diamond k}\|_{\mathcal{C}^{-\varepsilon}}^p\right] &\lesssim (p-1)^{p\frac{k}{2}} \\ &\quad + (p-1)^{p\frac{k}{2}} \int_{\mathcal{M}} \left(\int_{\mathbf{d}(x,y) \geq \mathbf{d}(x,z)} \mathbf{d}(x,y)^{\sigma-2} \mathbf{d}(x,z)^{\sigma-2} |\log(\mathbf{d}(x,y))|^k \, dy \, dz \right)^{\frac{p}{2}} dx. \end{aligned}$$

Since $|\log(\mathbf{d}(x,y))|^k \lesssim_k \mathbf{d}(x,y)^{-\frac{\sigma}{2}}$, we end up with

$$\mathbb{E}\left[\|{\stackrel{\bullet}{|}}_N^{\diamond k}\|_{\mathcal{C}^{-\varepsilon}}^p\right] \lesssim (p-1)^{p\frac{k}{2}} + (p-1)^{p\frac{k}{2}} \int_{\mathcal{M}} \left(\int_{\mathbf{d}(x,y) \geq \mathbf{d}(x,z)} \mathbf{d}(x,y)^{\frac{\sigma}{2}-2} \mathbf{d}(x,z)^{\sigma-2} \, dy \, dz \right)^{\frac{p}{2}} dx,$$

where the inner integral on the right-hand side is bounded uniformly in $x \in \mathcal{M}$ as $\mathcal{M}$ is 2-dimensional and compact. Using next Minkowski's integral inequality, we get

$$\mathbb{E}\left[\|{\stackrel{\bullet}{|}}^k\|_{L_T^q \mathcal{C}_x^{-\varepsilon}}^p\right] \leq \left\| \mathbb{E}\left[\|{\stackrel{\bullet}{|}}^k\|_{\mathcal{C}_x^{-\varepsilon}}^p\right]^{\frac{1}{p}} \right\|_{L_T^q}^p \lesssim (p-1)^{p\frac{k}{2}} T^{\frac{p}{q}}.$$

This gives the uniform boundedness of ${\stackrel{\bullet}{|}}_N^{\diamond k}$ in $L^p(\Omega; L^q([0,T]; \mathcal{C}^{-\varepsilon}(\mathcal{M})))$ for any $q \geq 1$. Chebychev's inequality and optimizing in p then ensures that the tail estimate (III.13) holds true:

$$\begin{aligned} \mathbb{P}\left(\|{\stackrel{\bullet}{|}}_N^{\diamond k}\|_{L_T^q \mathcal{C}_x^{-\varepsilon}} > R\right) &\leq C(p-1)^{p\frac{k}{2}} R^{-p} T^{\frac{p}{q}} \\ &\leq C e^{-cR^{\frac{2}{k}} T^{-\frac{2}{qk}}} \end{aligned}$$

for some constants c, C that do not depend on T, R, p, or N.
Now to show that $\{\Psi^{\diamond k}_N\}_N$ is Cauchy, let N_1, N_2 be two integers with $N_1 \leq N_2$, then as before we get

$$\begin{aligned}
&\mathbb{E}\left[\|\Psi^{\diamond k}_{N_1} - \Psi^{\diamond k}_{N_2}\|^p_{\mathcal{C}^{-\varepsilon}}\right]\\
&\lesssim \mathbb{E}\left[\|\Psi^{\diamond k}_{N_1} - \Psi^{\diamond k}_{N_2}\|^p_{W^{-\sigma,p}}\right]\\
&\lesssim \int_{\mathcal{M}} \mathbb{E}\left[\left((1-\Delta)^{\frac{-\sigma}{2}}(\Psi^{\diamond k}_{N_1} - \Psi^{\diamond k}_{N_2})(x)\right)^p\right] dx\\
&\lesssim (p-1)^{p\frac{k}{2}} \int_{\mathcal{M}} \mathbb{E}\left[\left((1-\Delta)^{\frac{-\sigma}{2}}(\Psi^{\diamond k}_{N_1} - \Psi^{\diamond k}_{N_2})(x)\right)^2\right]^{\frac{p}{2}} dx\\
&= (p-1)^{p\frac{k}{2}} \int_{\mathcal{M}} \left(\int_{\mathcal{M}^2} G_\sigma(x,y)G_\sigma(x,z)\mathbb{E}\left[(\Psi^{\diamond k}_{N_1} - \Psi^{\diamond k}_{N_2})(y)(\Psi^{\diamond k}_{N_1} - \Psi^{\diamond k}_{N_2})(z)\right] dy\, dz\right)^{\frac{p}{2}} dx
\end{aligned}$$

where we used Lemma III.4. Itô's isometry on the expectation term yields

$$\begin{aligned}
\mathbb{E}\left[(\Psi^{\diamond k}_{N_1} - \Psi^{\diamond k}_{N_2})(y)(\Psi^{\diamond k}_{N_1} - \Psi^{\diamond k}_{N_2})(z)\right] &= \mathbb{E}\left[\Psi^{\diamond k}_{N_1}(y)\Psi^{\diamond k}_{N_1}(z)\right] - \mathbb{E}\left[\Psi^{\diamond k}_{N_1}(y)\Psi^{\diamond k}_{N_2}(z)\right]\\
&\quad - \mathbb{E}\left[\Psi^{\diamond k}_{N_2}(y)\Psi^{\diamond k}_{N_1}(z)\right] + \mathbb{E}\left[\Psi^{\diamond k}_{N_2}(y)\Psi^{\diamond k}_{N_2}(z)\right]\\
&= G^{\mathcal{H}}_{N_1}(y,z)^k - G^{\mathcal{H}}_{N_1N_2}(y,z)^k - G^{\mathcal{H}}_{N_2N_1}(y,z)^k + G^{\mathcal{H}}_{N_2}(y,z)^k.
\end{aligned}$$

Using successively the mean value inequality, estimate (I.12), and again the comparison between logarithm and monomials, we get for any $0 < \delta \ll 1$:

$$\begin{aligned}
|G^{\mathcal{H}}_{N_1}(y,z)^k - G^{\mathcal{H}}_{N_1N_2}(y,z)^k| &\lesssim |G^{\mathcal{H}}_{N_1}(y,z) - G^{\mathcal{H}}_{N_1N_2}(y,z)|(1+\mathbf{d}(y,z)^{-\frac{\sigma}{6}})\\
&\lesssim ((|\log(\mathbf{d}(y,z)+N_1^{-1})| \vee 1) \wedge (N_1^{\delta-1}\mathbf{d}(y,z)^{-1}))(1+\mathbf{d}(y,z)^{-\frac{\sigma}{6}}).
\end{aligned}$$

In the case $\mathbf{d}(x,y) \leq N_1^{-1}$, we have by interpolation and the comparison between logarithm and monomials that

$$\begin{aligned}
((|\log(\mathbf{d}(y,z)+N_1^{-1})| \vee 1) \wedge (N_1^{\delta-1}\mathbf{d}(y,z)^{-1})) &\lesssim \log N_1 \wedge (N_1^{\delta-1}\mathbf{d}(y,z)^{-1}))\\
&\lesssim_{\delta,\theta} N_1^{(\frac{\delta}{2}-1)\theta}\mathbf{d}(y,z)^{-\theta}
\end{aligned}$$

for any $\theta \in (0;1)$, uniformly in N_1, while in the case $\mathbf{d}(x,y) \geq N_1^{-1}$ we get

$$\begin{aligned}
((|\log(\mathbf{d}(y,z)+N_1^{-1})| \vee 1) \wedge (N_1^{\delta-1}\mathbf{d}(y,z)^{-1})) &\lesssim \log \mathbf{d}(y,z)^{-1} \wedge (N_1^{\delta-1}\mathbf{d}(y,z)^{-1}))\\
&\lesssim_{\delta,\theta} N^{(\delta-1)\theta}\mathbf{d}(y,z)^{-((1-\theta)\frac{\nu}{2}+\theta)}
\end{aligned}$$

for any $\theta \in (0;1)$ and $0 < \nu \ll 1$, uniformly in N_1. This finally gives

$$|G^{\mathcal{H}}_{N_1}(y,z)^k - G^{\mathcal{H}}_{N_1N_2}(y,z)^k| \lesssim N_1^{-2\kappa}\mathbf{d}(y,z)^{-\frac{\sigma}{2}} \tag{III.15}$$

for some $\kappa > 0$ and implicit constant that is uniform in y, z and N_1, N_2. Similar computations as before yield that the whole inner integral on $\mathcal{M}^2$ is bounded uniformly in $x \in \mathcal{M}$ with

$$\int_{\mathcal{M}^2} G_\sigma(x,y)G_\sigma(x,z)\mathbb{E}\left[(\Psi_{N_1}^{\diamond k} - \Psi_{N_2}^{\diamond k})(y)(\Psi_{N_1}^{\diamond k} - \Psi_{N_2}^{\diamond k})(z)\right] dy\, dz \lesssim N_1^{-2\kappa}$$

so that, using Minkowski inequality, $(\Psi_N^{\diamond k})_N$ indeed is a Cauchy sequence in $L^p(\Omega; L^q([0,T]; \mathcal{C}^{-\varepsilon}(\mathcal{M})))$, we denote by $\Psi^{\diamond k}$ the limiting process.

Since $L^p(\Omega)$ convergence ensures convergence in probability, we can use Chebychev's inequality together with passing to the limit $N_2 \to +\infty$ and optimizing in p so that

$$\begin{aligned}
\mathbb{P}\left(\|\Psi_N^{\diamond k} - \Psi^{\diamond k}\|_{L_T^q\mathcal{C}_x^{-\varepsilon}} > N^{-\kappa}R\right) &\leq (p-1)^{p\frac{k}{2}} N^{\kappa p}R^{-p}\mathbb{E}\left[\|\Psi_N^{\diamond k} - \Psi^{\diamond k}\|^p_{L_T^q\mathcal{C}_x^{-\varepsilon}}\right] \\
&\leq C(p-1)^{p\frac{k}{2}}R^{-p}T^{\frac{p}{q}} \\
&\leq Ce^{-cR^{\frac{2}{k}}T^{-\frac{2}{qk}}}
\end{aligned}$$

for some constants $c, C > 0$ and the same exponent $\kappa > 0$ defined above. Note that c, C and κ do not depend on N, R or T. This proves deviation estimate (III.14) as well as almost sure convergence in $L^q([0,T], \mathcal{C}^{-\varepsilon}(\mathcal{M}))$ using Borel-Cantelli lemma.

Now for the continuity of $t \mapsto \Psi(t) \in \mathcal{C}^{-\varepsilon}(\mathcal{M})$, we use Kolmogorov's continuity criterion. We have similarly as above for some large p depending on ε, using Sobolev inequality and the Wiener chaos estimate

$$\begin{aligned}
&\mathbb{E}\Big\|\Psi(t+\delta) - \Psi(t)\Big\|^p_{\mathcal{C}^{-\varepsilon}} \lesssim \mathbb{E}\Big\|\Psi(t+\delta) - \Psi(t)\Big\|^p_{W^{-\frac{\varepsilon}{2},p}} \\
&\lesssim (p-1)^{\frac{p}{2}}\int_{\mathcal{M}}\left(\int_{\mathcal{M}^2} G_{\frac{\varepsilon}{2}}(x,y)G_{\frac{\varepsilon}{2}}(x,z)\mathbb{E}\Big[(\Psi(t+\delta,y) - \Psi(t,y))(\Psi(t+\delta,z) - \Psi(t,z))\Big]dydz\right)^{\frac{p}{2}}dx \\
&= (p-1)^{\frac{p}{2}}\int_{\mathcal{M}}\left(\int_{\mathcal{M}^2} G_{\frac{\varepsilon}{2}}(x,y)G_{\frac{\varepsilon}{2}}(x,z)\Big[(1-e^{-\delta\mathcal{H}})\otimes 1\Big]G^{\mathcal{H}}(y,z)dydz\right)^{\frac{p}{2}}dx.
\end{aligned}$$

Now, using Sobolev inequality,

$$\begin{aligned}
\Big\|\int_{\mathcal{M}} G_{\frac{\varepsilon}{2}}(x,y)\Big[(1-e^{-\delta\mathcal{H}})\otimes 1\Big]G^{\mathcal{H}}(y,z)dy\Big\|_{L^\infty_{x,z}} &= \Big\|\Big[(1-e^{-\delta\mathcal{H}})\otimes 1\Big]G^{\mathcal{H}}(x,z)\Big\|_{L_z^\infty W_x^{-\frac{\varepsilon}{2},\infty}} \\
&\lesssim \Big\|\Big[(1-e^{-\delta\mathcal{H}})\otimes 1\Big]G^{\mathcal{H}}(x,z)\Big\|_{L_z^\infty \mathcal{D}_x^{1-\frac{\varepsilon}{4}}},
\end{aligned}$$

with $\mathcal{D}^{1-\frac{\varepsilon}{4}}$ the Sobolev space associated to $\mathcal{H}$, see (I.7). Using the mean value inequality and the bound $|1-e^{-\delta\lambda_n}|^2 \lesssim (\delta\lambda_n)^{\frac{\varepsilon}{8}}$, for any fixed $z \in \mathcal{M}$,

$$\begin{aligned}
\Big\|\Big[(1-e^{-\delta\mathcal{H}})\otimes 1\Big]G^{\mathcal{H}}(\cdot,z)\Big\|^2_{\mathcal{D}_x^{1-\frac{\varepsilon}{4}}} &= \sum_{n\geq 0}\frac{|1-e^{-\delta\lambda_n}|^2}{\lambda_n^{1+\frac{\varepsilon}{4}}}\varphi_n(z)^2 \\
&\lesssim \delta^{\frac{\varepsilon}{8}}\sum_{n\geq 0}\lambda_n^{-1-\frac{\varepsilon}{8}}\varphi_n(z)^2
\end{aligned}$$

$$
\begin{aligned}
&\sim \delta^{\frac{\varepsilon}{8}}\|\delta_z\|^2_{\mathcal{D}^{-1-\frac{\varepsilon}{8}}}\\
&\lesssim \delta^{\frac{\varepsilon}{8}},
\end{aligned}
$$

uniformly in $z\in\mathcal{M}$, where we used that $\|\delta_z\|_{\mathcal{D}^{-1-\frac{\varepsilon}{8}}}$ is uniformly bounded in $z\in\mathcal{M}$, see the proof of [BDM23, Lemma 23]. Since $G_{\frac{\varepsilon}{2}}(x,z)\in L^\infty_x L^1_z$, and since $\mathcal{M}$ is compact, putting everything together we get

$$
\mathbb{E}\Big\|\text{⫯}(t+\delta)-\text{⫯}(t)\Big\|^p_{\mathcal{C}^{-\varepsilon}}\lesssim \delta^{p\frac{\varepsilon}{32}},
$$

from which $\text{⫯}\in C(\mathbb{R};\mathcal{C}^{-\varepsilon}(\mathcal{M}))$ a.s. due to Kolmogorov continuity criterion [3]. ▷

In our globalisation argument in Section III.3.3 below, we will also need to investigate the regularization of $\text{⫯}^{\diamond k}$ based on $\mathcal{H}$ instead of Δ.

Proposition III.8 – *Fix $k\in\mathbb{N}$ and $\chi\in C_0^\infty(\mathbb{R})$ supported in $(-1;1)$. For $M\in\mathbb{N}^*$, let $\chi_M=\chi(M^{-2}\mathcal{H})$. Then for any $\varepsilon\in(0,1)$, any $p\ge q\ge 1$, and any $N\in\mathbb{N}^*$, the sequence $\{(\mathbf{P}_N\chi_M\text{⫯})^{\diamond k}\}_M$ is a Cauchy sequence in $L^p(\Omega;L^q([0;T];\mathcal{C}^{-\varepsilon}(\mathcal{M})))$ that also converges almost surely in $L^q([0;T];\mathcal{C}^{-\varepsilon}(\mathcal{M}))$ to $(\mathbf{P}_N\text{⫯})^{\diamond k}$. Moreover, $(\mathbf{P}_N\chi_M\text{⫯})^{\diamond k}$ is uniformly bounded in N,M in $L^p(\Omega;L^q([0;T];\mathcal{C}^{-\varepsilon}(\mathcal{M})))$, and we have the tail estimates*

$$
\mathbb{P}\Big(\|(\mathbf{P}_N\chi_M\text{⫯})^{\diamond k}\|_{L^q_T\mathcal{C}^{-\varepsilon}_x}>R\Big)\le Ce^{-cR^{\frac{2}{k}}T^{-\frac{2}{qk}}} \tag{III.16}
$$

and

$$
\mathbb{P}\Big(\|(\mathbf{P}_N\chi_M\text{⫯})^{\diamond k}-(\mathbf{P}_N\text{⫯})^{\diamond k}\|_{L^q_T\mathcal{C}^{-\varepsilon}_x}>N^{c\kappa}M^{-\kappa}R\Big)\le Ce^{-cR^{\frac{2}{k}}T^{-\frac{2}{qk}}}, \tag{III.17}
$$

uniformly in R, p, and N,M.

Similar results hold for $(\mathbf{P}_N\chi_M\text{⫯}(0))^{\diamond k}$ in $\mathcal{C}^{-\varepsilon}(\mathcal{M})$.

Proof – Repeating the arguments above in the proof of Proposition III.7, we have for fixed $t\ge 0$:

$$
\begin{aligned}
&\mathbb{E}\Big[\|(\mathbf{P}_N\chi_{M_1}\text{⫯}(t))^{\diamond k}-(\mathbf{P}_N\chi_{M_2}\text{⫯}(t))^{\diamond k}\|^p_{\mathcal{C}^{-\varepsilon}}\Big]\\
&\lesssim_p \int_{\mathcal{M}}\Bigg(\int_{\mathcal{M}^2}G_{\frac{\varepsilon}{4}}(x,y)G_{\frac{\varepsilon}{4}}(x,z)\\
&\times\Big[G^{\mathcal{H}}_{N,M_1,M_1}(y,z)^k-G^{\mathcal{H}}_{N,M_1,M_2}(y,z)^k-G^{\mathcal{H}}_{N,M_2,M_1}(y,z)^k+G^{\mathcal{H}}_{N,M_2,M_2}(y,z)^k\Big]\,dy\,dz\Bigg)^{\frac{p}{2}}\,dx
\end{aligned}
$$

for some $p\gg 1$ depending on ε, and where

$$
G^{\mathcal{H}}_{N,M_j,M_\ell}=(\mathbf{P}_N\chi_{M_j}\otimes\mathbf{P}_N\chi_{M_\ell})G^{\mathcal{H}}=(\mathbf{P}_N\otimes\mathbf{P}_N)(\chi_{M_j}\otimes\chi_{M_\ell})G^{\mathcal{H}}.
$$

Then using the mean value theorem and (I.17)-(I.18)-(A.12) we can estimate the

3. Kolmogorov regularity theorem rather ensures that there exists a modification of the process ⫯_N (resp. ⫯) that is almost surely continuous. We identify ⫯_N (resp. ⫯) with this continuous modification in our argument.

inner integrals by

$$\int_{\mathcal{M}^2} \mathbf{d}(x,y)^{\frac{\varepsilon}{4}-2}\mathbf{d}(x,y)^{\frac{\varepsilon}{4}-2}N^{c\delta}M_1^{-\delta}|\log(\mathbf{d}(y,z))|^{k-1}dydz \lesssim N^{c\delta}M_1^{-\delta}$$

for some small $\delta > 0$. This is enough for (III.17) after finishing the proof as in that of Proposition III.7.
The estimate (III.16) follows similarly, using that we have a bound uniform in $N \in \mathbb{N}^*$ from using only (I.17) and not (I.18). ▷

III.2.3 Truncated Gibbs measure and convergence

We finally construct the Gibbs measure ρ as the limit in total variation of $(\rho_N)_N$ given by (III.8):

$$d\rho_N(u) = \mathcal{Z}_N^{-1}e^{-\int_{\mathcal{M}} F_N^{\diamond}(\mathbf{P}_N u)\,dx}\,d\mu^{\mathcal{H}}(u),$$

where $\mathcal{Z}_N$ is the normalization constant

$$\mathcal{Z}_N := \mathbb{E}_{\mu^{\mathcal{H}}}\left[e^{-\int_{\mathcal{M}} F_N^{\diamond}(\mathbf{P}_N u)\,dx}\right].$$

First, we prove the convergence of the density functions.

Proposition III.9 – *For any $p \geq 1$, the sequence $\left(\int_{\mathcal{M}} F_N^{\diamond}(\mathbf{P}_N u)\,dx\right)_N$ is a Cauchy sequence in $L^p(\mu^{\mathcal{H}})$, converging to a limit denoted by $\int_{\mathcal{M}} F^{\diamond}(u)\,dx \in L^p(\mu^{\mathcal{H}})$.*

Proof – Since F is a polynomial of degree $k = 2m$, it is sufficient to treat the case where $F(X) = X^k$ is a monomial. The proof follows the same lines as the one of Proposition III.7. First, we have the bound

$$\begin{aligned}
\left\|\int_{\mathcal{M}} (\mathbf{P}_N u)^{\diamond k}\,dx\right\|_{L^p(\mu^{\mathcal{H}})}^2 &\lesssim_p \left\|\int_{\mathcal{M}} (\mathbf{P}_N u)^{\diamond k}\,dx\right\|_{L^2(\mu^{\mathcal{H}})}^2 \\
&= \int_{\mathcal{M}^2} \mathbb{E}\left[(\mathbf{P}_N u)^{\diamond k}(x)(\mathbf{P}_N u)^{\diamond k}(y)\right]\,dx\,dy \\
&= \int_{\mathcal{M}^2} \mathbb{E}\left[(\mathbf{P}_N u)(x)(\mathbf{P}_N u)(y)\right]^k\,dx\,dy \\
&= \int_{\mathcal{M}^2} G_N^{\mathcal{H}}(x,y)^k\,dx\,dy \\
&\lesssim \int_{\mathcal{M}^2} \Big(1 + |\log(\mathbf{d}(x,y) + N^{-1})|\Big)^k\,dx\,dy \\
&\lesssim 1,
\end{aligned}$$

uniformly in N, where we used successively the Wiener chaos estimate (III.6), the algebraic rule (III.4) and the estimate (I.9). This proves the uniform boundedness in $L^p(\mu^{\mathcal{H}})$.
Now to prove the Cauchy property, take $N_2 \geq N_1 \geq 1$ and following the lines of

Proposition III.7, we have

$$
\begin{aligned}
&\left\| \int_{\mathcal{M}} \Big((\mathbf{P}_{N_1} u)^{\diamond k} - (\mathbf{P}_{N_2} u)^{\diamond k} \Big) \, dx \right\|_{L^p(\mu^{\mathcal{H}})}^2 \\
&\lesssim_p \left\| \int_{\mathcal{M}} \Big((\mathbf{P}_{N_1} u)^{\diamond k} - (\mathbf{P}_{N_2} u)^{\diamond k} \Big) \, dx \right\|_{L^2(\mu^{\mathcal{H}})}^2 \\
&= \int_{\mathcal{M}^2} \mathbb{E}\Big[\Big((\mathbf{P}_{N_1} u)^{\diamond k} - (\mathbf{P}_{N_2} u)^{\diamond k} \Big)(x) \Big((\mathbf{P}_{N_1} u)^{\diamond k} - (\mathbf{P}_{N_2} u)^{\diamond k} \Big)(y) \Big] \, dx \, dy \\
&= \int_{\mathcal{M}^2} \Big(G^{\mathcal{H}}_{N_1}(x,y)^k - G^{\mathcal{H}}_{N_1 N_2}(x,y)^k - G^{\mathcal{H}}_{N_2 N_1}(x,y)^k + G^{\mathcal{H}}_{N_2}(x,y)^k \Big) \, dx \, dy.
\end{aligned}
$$

From there, the estimate (III.15) from the proof of Proposition III.7 ensures that the right-hand side vanishes as $N_1 \to +\infty$, proving that $\left(\int_{\mathcal{M}} F_N^{\diamond}(\mathbf{P}_N u) \, dx \right)_N$ is a Cauchy sequence in $L^p(\mu^{\mathcal{H}})$. ▷

Write $R_N(u) := e^{-\int_{\mathcal{M}} F_N^{\diamond}(\mathbf{P}_N u)\, dx}$ for the exponential density function. Note that as a direct consequence of Proposition III.9, we get the convergence in measure of $(R_N(u))_N$ with respect to $\mu^{\mathcal{H}}$ as a continuous function of the convergent sequence $\left(- \int_{\mathcal{M}} F_N^{\diamond}(\mathbf{P}_N u) \, dx \right)_N$. To conclude that the convergence also happens in $L^p(\mu^{\mathcal{H}})$ for $p \geq 1$, fix $N_1, N_2 \geq 1$ and $p \geq 1$. Denote by $A_{N_1,N_2,\eta}$ the event

$$
A_{N_1,N_2,\eta} := \{ |R_{N_1}(u) - R_{N_2}(u)| > \eta \}
$$

then convergence in measure of $R_N(u)$ ensures that for any $\eta > 0$,

$$
\mu^{\mathcal{H}}(A_{N_1,N_2,\eta}) \to 0 \quad \text{as} \quad N_1, N_2 \to +\infty.
$$

Thus

$$
\begin{aligned}
\|R_{N_1}(u) - R_{N_2}(u)\|_{L^p(\mu^{\mathcal{H}})} &\leq \left\| (R_{N_1}(u) - R_{N_2}(u)) \, \mathbf{1}_{A_{N_1,N_2,\eta}} \right\|_{L^p(\mu^{\mathcal{H}})} \\
&\quad + \left\| (R_{N_1}(u) - R_{N_2}(u)) \, \mathbf{1}_{A^c_{N_1,N_2,\eta}} \right\|_{L^p(\mu^{\mathcal{H}})} \\
&\leq \mu^{\mathcal{H}}(A_{N_1,N_2,\eta})^{\frac{1}{2p}} \, \|R_{N_1}(u) - R_{N_2}(u)\|_{L^{2p}(\mu^{\mathcal{H}})} + \mu^{\mathcal{H}}(A^c_{N_1,N_2,\eta})^{\frac{1}{p}} \eta.
\end{aligned}
\tag{III.18}
$$

From there, we see that a uniform bound on $R_N(u)$ in $L^{2p}(\mu^{\mathcal{H}})$ is enough to get convergence of $R_N(u)$ in $L^p(\mu^{\mathcal{H}})$. This will be provided by the Boué-Dupuis' formula in the form introduced in [Üst14] and [BG23]. Roughly speaking, given a cylindrical Brownian motion $X = (X_t)_{t \in [0,1]}$ on $L^2(\mathcal{M})$ and under some assumptions on the non-linear functional $J : C([0,1]; C^{(-1)^-}(\mathcal{M})) \to \mathbb{R}$, then the following variational formula holds:

$$
\log \mathbb{E}\left[e^{-J(X)} \right] = \sup_v \mathbb{E}\left[-J\left(X + \int_0^{\cdot} v(t) \, dt \right) - \frac{1}{2} \int_0^1 \|v(t)\|^2_{L^2(\mathcal{M})} \, dt \right], \tag{III.19}
$$

where the supremum is taken over all the progressively measurable processes $v \in L^2(\Omega; L^2([0,1]; L^2(\mathcal{M})))$. This allows to bound uniformly the partition function $\mathcal{Z}_N$.

Proposition III.10 – *The normalization constant $\mathcal{Z}_N = \mathbb{E}_{\mu^{\mathcal{H}}}[R_N(u)]$ is uniformly bounded :*

$$\sup_N \mathbb{E}_{\mu^{\mathcal{H}}}[R_N(u)] < +\infty.$$

Proof – The assumption on J is to be *tame* (see [BG23, Definition 1]) with respect to the law of X for which an easy sufficient condition is to be bounded below and have some finite moment. Consider therefore the functional

$$J_N(X) := \int_{\mathcal{M}} F_N^{\diamond}(\mathbf{P}_N \mathcal{H}^{-\frac{1}{2}} X_{t=1})\, dx.$$

First of all note that J_N is well defined and measurable on $C([0,1]; \mathcal{C}^{-1-\kappa}(\mathcal{M}))$ by composition, for any $\kappa > 0$. Since by assumption F is a polynomial of even degree and positive leading coefficient, so is $F_N^{\diamond}$ and therefore J_N is bounded from below for each fixed N.
Then, note that of $u = \mathcal{H}^{-\frac{1}{2}} X_{t=1}$ has law given by the GFF $\mu^{\mathcal{H}}$ in (III.4), for which we already proved that

$$\int_{\mathcal{M}} F_N^{\diamond}(\mathbf{P}_N u)\, dx$$

is bounded uniformly in $L^p(\mu^{\mathcal{H}})$, see Proposition III.9. Therefore, the functional J_N is tame in the sense of [BG23], that is

$$\mathbb{E}_{\mathbb{P}}\left[e^{-qJ}\right] + \mathbb{E}_{\mathbb{P}}[|J|^p] < +\infty$$

for some conjugate exponents $p, q > 1$, and Boué-Dupuis' formula holds true in our context. As the supremum runs over the progressively measurable processes $v \in L^2(\mathbb{P}; L^2([0,1]; L^2(\mathcal{M})))$, we define the process Θ by

$$\Theta(t) := \int_0^t \mathcal{H}^{-\frac{1}{2}} v(s)\, ds,$$

so that $\dot{\Theta} = \mathcal{H}^{-\frac{1}{2}} v$ and identity (III.19) writes

$$\log \mathbb{E}_{\mu^{\mathcal{H}}}[R_N(u)] = \sup_{\Theta} \mathbb{E}\left[-\int_{\mathcal{M}} F_N^{\diamond}\left(\mathbf{P}_N u + \mathbf{P}_N \Theta(1)\right)\, dx - \frac{1}{2}\int_0^1 \|\dot{\Theta}\|^2_{D(\sqrt{\mathcal{H}})}\, dt\right]. \tag{III.20}$$

Even though F has positive leading coefficient, $F_N^{\diamond}$ is not bounded from below uniformly in N. We thus expand the non-linearity using the binomial rule in Lemma III.5 to get

$$F_N^{\diamond}\left(\mathbf{P}_N u + \mathbf{P}_N \Theta(1)\right) = \sum_{p=0}^{2m} a_p H_p(\mathbf{P}_N u + \mathbf{P}_N \Theta(1), \sigma_N^2)$$

$$
\begin{aligned}
&= \sum_{p=0}^{2m}\sum_{j=0}^{p} a_p \binom{p}{j} (\mathbf{P}_N u)^{\diamond j} (\mathbf{P}_N \Theta(1))^{p-j} \qquad \text{(III.21)}\\
&= F(\mathbf{P}_N \Theta(1)) + \sum_{p=1}^{2m}\sum_{j=1}^{p} a_p \binom{p}{j} (\mathbf{P}_N u)^{\diamond j} (\mathbf{P}_N \Theta(1))^{p-j}.
\end{aligned}
$$

The assumption that F is bounded below is crucial here as the contribution from $j=0$ together with the quadratic part in (III.20),

$$
-\int_{\mathcal{M}} F(\mathbf{P}_N\Theta(1))\,dx - \frac{1}{2}\int_0^1 \|\dot{\Theta}\|^2_{D(\sqrt{\mathcal{H}})}\,dt,
$$

will control all the cross product terms in between. The duality pairing between $\mathcal{C}^{-\varepsilon}$ and $B^{\varepsilon}_{1,1} \supset W^{\varepsilon,1}$ yields

$$
\left|\int_{\mathcal{M}} (\mathbf{P}_N u)^{\diamond j} (\mathbf{P}_N \Theta(1))^{p-j}\,dx\right| \lesssim \|(\mathbf{P}_N u)^{\diamond j}\|_{\mathcal{C}^{-\varepsilon}(\mathcal{M})} \|(\mathbf{P}_N\Theta(1))^{p-j}\|_{W^{\varepsilon,1}(\mathcal{M})}.
$$

At this point, the only remaining difficulty is to deal with the terms involving Θ in (III.21). For the sake of clarity in the following estimates, let v be a generic element of $W^{\varepsilon,1}(\mathcal{M})$. First use the interpolation inequality

$$
\|v\|_{W^{\varepsilon,1}} \lesssim \|v\|_{W^{1-\eta,1}}^{\frac{\varepsilon}{1-\eta}} \|v\|_{L^1}^{\frac{1-\eta-\varepsilon}{1-\eta}}
$$

for $\eta \in (0,1)$ small enough. Together with Young's inequality, this yields

$$
\|v\|_{W^{\varepsilon,1}} \lesssim \|v\|_{W^{1-\eta,1}}^{\frac{\varepsilon}{1-\eta}r} + \|v\|_{L^1}^{\frac{1-\eta-\varepsilon}{1-\eta}r'}
$$

for some $\frac{1}{r}+\frac{1}{r'} = 1$ to be chosen later. Recall the fractional Leibniz rule from Lemma A.6:

$$
\|uv\|_{H^t} \lesssim \|u\|_{H^s}\|v\|_{H^r}
$$

as long as $t = r+s-1$ satisfy $0 < r+s$ and $r,s<1$. Then, taking $r = s = 1-\frac{\eta}{2}$, we get

$$
\begin{aligned}
\|v^{p-j}\|_{H^{1-\eta}} &\lesssim \|v^{p-j-1}\|_{H^{1-\frac{\eta}{2}}} \|v\|_{H^{1-\frac{\eta}{2}}}\\
&\lesssim_j \|v\|^{p-j}_{H^{1-\frac{\eta}{2^{p-j}}}}
\end{aligned}
$$

and combined with the embedding $H^{1-\eta}(\mathcal{M}) \hookrightarrow W^{1-\eta,1}(\mathcal{M})$ and $L^{2m}(\mathcal{M}) \hookrightarrow L^{p-j}(\mathcal{M})$ as $\mathcal{M}$ is compact together with the fact that the form domain $D(\sqrt{\mathcal{H}})$ embeds in any $H^{1-\kappa}(\mathcal{M})$, we get

$$
\|v^{p-j}\|_{W^{\varepsilon,1}} \lesssim_j \|v\|_{H^{1-\frac{\eta}{2^{p-j}}}}^{\frac{\varepsilon}{1-\eta}(p-j)r} + \|v^{p-j}\|_{L^1}^{\frac{1-\eta-\varepsilon}{1-\eta}r'}
$$

$$\lesssim_j \|v\|_{D(\sqrt{\mathcal{H}})}^{\frac{\varepsilon}{1-\eta}(p-j)r} + \|v\|_{L^{2m}}^{\frac{1-\eta-\varepsilon}{1-\eta}(p-j)r'}.$$

Note that $H^{1-}(\mathcal{M})$ itself is not an algebra, hence the loss of regularity when performing the product estimate. However, Leibniz rule allows to still have a nice estimate in spaces below $H^1(\mathcal{M})$ and in the end, an estimate involving only the norm on $D(\sqrt{\mathcal{H}})$.

Recall that the goal is to deal with the supremum over Θ, hence the terms involving u are harmless as they are bounded in $L^p(\Omega, \mathcal{C}^{-\varepsilon}(\mathcal{M}))$ for any $p > 1, \varepsilon > 0$ and thus provide bounds uniformly in N. For $\delta > 0$ and $s > 1$, we use once again Young's inequality to have

$$\begin{aligned}
\left| \int_{\mathcal{M}} (\mathbf{P}_N u)^{\diamond j} (\mathbf{P}_N \Theta(1))^{p-j} \, dx \right| &\lesssim \|(\mathbf{P}_N u)^{\diamond j}\|_{\mathcal{C}^{-\varepsilon}(\mathcal{M})} \|(\mathbf{P}_N \Theta(1))^{p-j}\|_{W^{\varepsilon,1}(\mathcal{M})} \\
&\lesssim \|(\mathbf{P}_N u)^{\diamond j}\|_{\mathcal{C}^{-\varepsilon}(\mathcal{M})} \|\Theta(1)\|_{D(\sqrt{\mathcal{H}})}^{\frac{\varepsilon}{1-\eta}(p-j)r} \\
&\qquad + \|(\mathbf{P}_N u)^{\diamond j}\|_{\mathcal{C}^{-\varepsilon}(\mathcal{M})} \|\mathbf{P}_N \Theta(1)\|_{L^{2m}}^{\frac{1-\eta-\varepsilon}{1-\eta}(p-j)r'} \\
&\le C(\delta) \|(\mathbf{P}_N u)^{\diamond j}\|_{\mathcal{C}^{-\varepsilon}(\mathcal{M})}^{\frac{s}{s-1}} \\
&\qquad + \delta^s \|\Theta(1)\|_{D(\sqrt{\mathcal{H}})}^{\frac{\varepsilon}{1-\eta}(p-j)rs} + \delta^s \|\mathbf{P}_N \Theta(1)\|_{L^{2m}}^{\frac{1-\eta-\varepsilon}{1-\eta}(p-j)r's}.
\end{aligned}$$

Note that we dropped the regularization $\mathbf{P}_N$ in the $\|\mathbf{P}_N \Theta(1)\|_{H^{1-\frac{\eta}{2p-j}}}$ term as it maps usual Sobolev spaces to themselves, uniformly in N. In view of (III.20), to prove uniform boundedness we only need to tune the parameters so that the $D(\sqrt{\mathcal{H}})$ norm (resp. L^{2m}) is at most to the power 2 (resp. $2m$), hence the following conditions

$$\begin{aligned}
\frac{\varepsilon}{1-\eta}(p-j)r &< 2, \\
\frac{1-\eta-\varepsilon}{1-\eta}(p-j)r' &< 2m.
\end{aligned}$$

As $p - j < 2m$ for $j \neq 0$, the second condition is satisfied provided r is large enough. For such r being fixed, we just need to take ε small enough so that the first condition is satisfied as well. This ensures that for some $s > 1$ close enough to 1, the Young estimate above, together with δ chosen small enough, make the cross product terms be absorbed by

$$-\int_{\mathcal{M}} F(\mathbf{P}_N \Theta(1)) \, dx - \frac{1}{2} \int_0^1 \|\dot{\Theta}\|_{D(\sqrt{\mathcal{H}})}^2 \, dt.$$

Indeed, gathering everything together we get

$$\begin{aligned}
-\int_{\mathcal{M}} F_N^{\diamond} \left(\mathbf{P}_N u + \mathbf{P}_N \Theta(1) \right) \, dx - \frac{1}{2} \int_0^1 \|\dot{\Theta}\|_{D(\sqrt{\mathcal{H}})}^2 \, dt \\
= -\int_{\mathcal{M}} F(\mathbf{P}_N \Theta(1)) \, dx - \frac{1}{2} \int_0^1 \|\dot{\Theta}\|_{D(\sqrt{\mathcal{H}})}^2 \, dt
\end{aligned}$$

$$\begin{aligned}
&\quad - \sum_{p=1}^{2m} \sum_{j=1}^{p} a_p \binom{p}{j} \int_{\mathcal{M}} (\mathbf{P}_N u)^{\diamond j} (\mathbf{P}_N \Theta(1))^{p-j}\, dx \\
&\leq - \int_{\mathcal{M}} F(\mathbf{P}_N \Theta(1))\, dx - \frac{1}{2} \int_0^1 \|\dot{\Theta}\|^2_{D(\sqrt{\mathcal{H}})}\, dt \\
&\quad + \delta^s \|\Theta(1)\|^2_{D(\sqrt{\mathcal{H}})} + \delta^s \|\mathbf{P}_N \Theta(1)\|^{2m}_{L^{2m}} + C(\delta) \|(\mathbf{P}_N u)^{\diamond j}\|^{\frac{s}{s-1}}_{\mathcal{C}^{-\varepsilon}(\mathcal{M})} \\
&\leq - \int_{\mathcal{M}} \Big(F(\mathbf{P}_N \Theta(1)) - \delta^s (\mathbf{P}_N \Theta(1))^{2m} \Big)\, dx \\
&\quad - \Big(\frac{1}{2} - \delta^s \Big) \int_0^1 \|\dot{\Theta}\|^2_{D(\sqrt{\mathcal{H}})}\, dt + C(\delta) \|(\mathbf{P}_N u)^{\diamond j}\|^{\frac{s}{s-1}}_{\mathcal{C}^{-\varepsilon}(\mathcal{M})}.
\end{aligned}$$

Taking δ small enough ensures that the first two terms are bounded from above by a deterministic constant, uniformly in N and Θ. Taking the expectation, the last term is harmless as u is distributed according to $\mu^{\mathcal{H}}$ for which we already proved uniform bounds for the renormalized polynomials, this concludes the proof. ▷

Note that from the definition of R_N, then $(R_N)^p$ only amounts for $p \geq 1$ to scaling the non linear functional F by a factor p so that the above argument still holds for any order moment and we have

$$\sup_N \|R_N(u)\|_{L^p(\mu^{\mathcal{H}})} < +\infty.$$

Combined with estimate (III.18) we get the following proposition.

Proposition III.11 – *For any $p \geq 1$, $(R_N)_N$ is a Cauchy sequence in $L^p(\mu^{\mathcal{H}})$.*

We can now give the proof of our first result.

Proof of Theorem III.1 – Since for all $N \in \mathbb{N}^*$, the measures ρ_N are all defined by their density $\mathcal{Z}_N^{-1} R_N$ with respect to the same measure $\mu^{\mathcal{H}}$, and those density converge in $L^1(\mu^{\mathcal{H}})$, it ensures the convergence in total variation of $(\rho_N)_N$ to some measure ρ that is absolutely continuous with respect to $\mu^{\mathcal{H}}$. ▷

For our almost sure globalization argument, we will also need a bound uniform with respect to both a truncation with respect to Δ and one with respect to $\mathcal{H}$.

Proposition III.12 – *Let $\chi \in C_0^\infty(\mathbb{R})$ be supported in $(-1;1)$, and for $M \in \mathbb{N}^*$ set $\chi_M = \chi(M^{-2}\mathcal{H})$. Then*

$$\sup_{N,M} \mathbb{E}_{\mu^{\mathcal{H}}} [R_N(\chi_M u)] < +\infty.$$

Proof – The proof follows from repeating verbatim that of Proposition III.10 up to replacing Θ and u by $\chi_M \Theta$ and $\chi_M u$, and using that χ_M is uniformly bounded on $\mathcal{D}(\sqrt{\mathcal{H}})$, and the last part of Proposition III.8 instead of Proposition III.9. ▷

III.3 – Well-posedness theory

Now that the measure ρ has been constructed, we study a parabolic dynamics leaving it invariant. Similarly to ρ, the dynamics will rather be defined at a truncated level N and then passed to the limit. We therefore investigate the truncated dynamics (III.9) that we recall here

$$\partial_t u_N + \mathcal{H} u_N + \mathbf{P}_N f_N^{\diamond}(\mathbf{P}_N u_N) = \sqrt{2}\zeta$$

with $u(0) = u_0$. Note that we decide to truncate the non-linearity rather than the noise and initial data. This is mainly because the spectral projector $\mathbf{P}_N$ does not commute with $\mathcal{H}$; see Remark 5.
In this section we prove convergence of the smoothened dynamics (III.9) to a dynamics leaving ρ invariant, as well as deterministic local well-posedness and probabilistic global well-posedness results.

III.3.1 Construction of the dynamics

We study the dynamics

$$\begin{cases} \partial_t u_N + \mathcal{H} u_N + \mathbf{P}_N f_N^{\diamond}(\mathbf{P}_N u_N) = \sqrt{2}\zeta, \\ u_N(0) = u_0, \end{cases}$$

for some rough initial data $u_0 \in \mathcal{C}^{-\varepsilon}(\mathcal{M})$. As we investigate the invariance of the measure, u_0 is intended to be distributed with law ρ. Since ρ is absolutely continuous with respect to $\mu^{\mathcal{H}}$ which is supported in $\mathcal{C}^{-\varepsilon}$, it is enough to treat the case where $u_0 \in \mathcal{C}^{-\varepsilon}$.
We follow the usual Da Prato-Debussche trick from [DD03]: split the equation into a linear dynamics driven by ζ and a non-linear dynamics starting from zero initial data. To that end we write $u_N = \Psi + w_N$ where Ψ is the stationary solution to the linear equation

$$(\partial_t + \mathcal{H})\Psi = \sqrt{2}\zeta$$

and w_N solves the non-linear equation

$$\begin{cases} \partial_t w_N + \mathcal{H} w_N + \mathbf{P}_N f_N^{\diamond}(\mathbf{P}_N w_N + \mathbf{P}_N \Psi) = 0, \\ w_N(0) = u(0) - \Psi(0). \end{cases} \tag{III.22}$$

Note that the equation on w_N still contains stochastic forcing terms due to the appearance of Ψ in the non-linearity. Since Ψ contains the rough part of the solution u_N, $w_N = u_N - \Psi$ is expected to be smoother in some appropriate sense. In particular Proposition III.7 ensures that for any $p \geq q \geq 1$, $T > 0$ and $\varepsilon > 0$, $(\mathbf{P}_N \Psi)_N$ forms a Cauchy sequence in $L^p(\Omega; L^q([0;T]; \mathcal{C}^{-\varepsilon}(\mathcal{M})))$ that converges to Ψ. Moreover, Ψ is almost surely in $C([0,T]; \mathcal{C}^{-\varepsilon}(\mathcal{M}))$ and satisfies the tail and deviation estimates (III.13) and (III.14).
Recall that $f = F'$ where F is some polynomial of even degree $2m$ and positive leading coefficient. In particular, we write

$$f = \sum_{p=0}^{2m-1} b_p X^p$$

and can expand the non linear term in (III.22) using the algebraic rule (III.4):

$$f_N^{\diamond}(\mathbf{P}_N w_N + \mathbf{P}_N \overset{\bullet}{\vert}) = \sum_{p=0}^{2m-1} \sum_{j=0}^{p} b_p \binom{p}{j} (\mathbf{P}_N w_N)^j (\mathbf{P}_N \overset{\bullet}{\vert})^{\diamond(p-j)}.$$

For the sake of clarity in the following sections, we define

$$\mathbf{z}_N = \left(\mathbf{z}_N^{(j)}\right)_{0\le j\le 2m-1} := \left((\mathbf{P}_N \overset{\bullet}{\vert})^{\diamond j}\right)_{0\le j\le 2m-1}$$

to be the vector containing the renormalized powers of $\overset{\bullet}{\vert}_N$. We also define $\mathbf{z}$ the same way, dropping the smoothing operator $\mathbf{P}_N$. Note that it immediately follows that $(\mathbf{z}_N)_N$ converges to $\mathbf{z}$ in any $L^p(\Omega)$, as a $L^q([0;T];\mathcal{C}^{-\varepsilon}(\mathcal{M}))^{2m}$ valued random variable, for any $p, q \geq 1$.
We also define the corresponding non-linearity

$$\mathbf{f}_N(w, \mathbf{z}) := \sum_{p=0}^{2m-1} \sum_{j=0}^{p} b_p \binom{p}{j} (\mathbf{P}_N w)^j \mathbf{z}^{(p-j)}, \tag{III.23}$$

and $\mathbf{f}$ in the same way dropping the operator $\mathbf{P}_N$, so that we are left with the study of the equation

$$\begin{cases} \partial_t w_N + \mathcal{H} w_N + \mathbf{P}_N \mathbf{f}_N(w_N, \mathbf{z}_N) = 0, \\ w_N(0) = u_0 - \overset{\bullet}{\vert}(0). \end{cases} \tag{III.24}$$

Provided we can solve (III.22) in an appropriate functional space, solutions to (III.9) will therefore be constructed as $u_N = \overset{\bullet}{\vert} + w_N$. Note that $\overset{\bullet}{\vert}$ in the decomposition does not depend on N, this is because we study the dynamics where neither the noise nor the initial condition are smoothened, but only the non-linear term.

III.3.2 Deterministic local well-posedness

We investigate the well-posedness of (III.9) for deterministic initial data $u_0 \in \mathcal{C}^{-\varepsilon}(\mathcal{M})$. In that case, we can postulate

$$u_N = \overset{\bullet}{\vert} + w_N,$$

where w_N solves the non-linear equation

$$\begin{cases} \partial_t w_N + \mathcal{H} w_N + \mathbf{P}_N f_N^{\diamond}(\mathbf{P}_N w_N + \overset{\bullet}{\vert}_N) = 0, \\ w_N(0) = u_0 - \overset{\bullet}{\vert}(0). \end{cases} \tag{III.25}$$

In view of the Schauder estimate (I.6) for $\mathcal{H}$, and the initial condition belonging to

$\mathcal{C}^{-\varepsilon}(\mathcal{M})$, one expects w_N to belong to the following space:

$$X_T^{-\varepsilon,\sigma} = C([0,T],\mathcal{C}^{-\varepsilon}(\mathcal{M})) \cap C((0,T],\mathcal{C}^{\sigma}(\mathcal{M})), \tag{III.26}$$

endowed with the norm

$$\|w\|_{X_T^{-\varepsilon,\sigma}} = \|w\|_{L_T^\infty \mathcal{C}^{-\varepsilon}} + \sup_{0<t\le T} t^{\frac{\sigma+\varepsilon}{2}} \|w(t)\|_{\mathcal{C}^\sigma},$$

for $\varepsilon < \sigma < 1$ that takes account of the blow-up at $t=0$. The unusual restriction $\sigma < 1$ instead of $\sigma < 2$ in the case of the standard Laplace operator is due to the roughness of the Anderson Hamiltonian. One can prove that the solution belongs to the L^2-based Sobolev space $\mathcal{D}^{2-}$ associated to $\mathcal{H}$ which, only embeds in C^{1-}. Note that even if (III.25) is a truncated equation, as $\mathbf{P}_N$ maps Sobolev spaces to themselves, it falls into the scope of the following more general problem

$$\begin{cases} \partial_t w + \mathcal{H}w + \mathbf{f}(w,\mathbf{z}) = 0, \\ w(0) = w_0, \end{cases} \tag{III.27}$$

for the non-linearity $\mathbf{f}$ described above, a vector $\mathbf{z} \in L^q([0;T];\mathcal{C}^{-\varepsilon}(\mathcal{M}))^{2m}$ and initial condition $w_0 \in \mathcal{C}^{-\varepsilon}(\mathcal{M})$. We then have the following local well-posedness result.

Proposition III.13 – *Let $0 < \varepsilon < \sigma < 1$ and $1 \le q < \infty$ be such that $\frac{\sigma+\varepsilon}{2}\frac{q}{q-1} < \frac{1}{2m-1}$. Then there exists $\theta, C > 0$ such that for any $T_0, r, R > 0$, there exists $T \propto \min\Big(T_0; \Big(\frac{r}{(1+r+R)^{2m-1}}\Big)^\theta\Big) \in (0; T_0 \wedge 1]$ such that for any $w_0 \in \mathcal{C}^{-\varepsilon}(\mathcal{M})$ and $\mathbf{z} \in L^q([0;T_0],\mathcal{C}^{-\varepsilon}(\mathcal{M}))^{2m}$ satisfying $\|w_0\|_{\mathcal{C}^{-\varepsilon}} \le r$ and $\|\mathbf{z}\|_{L^q_{T_0}\mathcal{C}^{-\varepsilon}} \le R$, there exists a unique solution w in $X_T^{-\varepsilon,\sigma}$ to (III.27). Moreover w depends Lipschitz continuously on both w_0 and $\mathbf{z}$, and satisfies*

$$\|w\|_{X_T^{-\varepsilon,\sigma}} \le Cr. \tag{III.28}$$

Proof – For $T \in (0;1]$, we define the solution mapping $\Phi_{(w_0,\mathbf{z})}$ by

$$\Phi_{(w_0,\mathbf{z})}(w)(t) := e^{-t\mathcal{H}} w_0 - \int_0^t e^{-(t-s)\mathcal{H}} \mathbf{f}(w,\mathbf{z})(s)\, ds$$

and seek for a solution w as a fixed point of $\Phi_{(w_0,\mathbf{z})}$ in the ball of radius $4Cr$ in $X_T^{-\varepsilon,\sigma}$ for some $C > 0$ and small enough $T > 0$ depending on r, R as in the statement of Proposition III.13.
First, for v in the ball of radius $4Cr$ in $X_T^{-\varepsilon,\sigma}$, using the expression (III.23) of $\mathbf{f}$, the Schauder estimate (I.6) for $\mathcal{H}$, and a repeated use of the product estimates in Lemma A.3 (iii), we estimate

$$\Big\|\Phi_{(w_0,\mathbf{z})}(w)\Big\|_{L_T^\infty \mathcal{C}^{-\varepsilon}} \lesssim \|w_0\|_{\mathcal{C}^{-\varepsilon}} + \int_0^T \Big\|\mathbf{f}(w,\mathbf{z})(s)\Big\|_{\mathcal{C}^{-\varepsilon}} ds$$

$$
\begin{aligned}
&\lesssim \|w_0\|_{\mathcal{C}^{-\varepsilon}} + \sum_{p=0}^{2m-1}\sum_{j=0}^{p}|b_p|\int_0^T \left\|w^j \mathbf{z}^{(p-j)}\right\|_{\mathcal{C}^{-\varepsilon}} ds \\
&\lesssim \|w_0\|_{\mathcal{C}^{-\varepsilon}} + \sum_{p=0}^{2m-1}\Big\{\sum_{j=1}^{p}|b_p|\int_0^T \|w(s)\|_{\mathcal{C}^{\sigma}}^j \|\mathbf{z}^{(p-j)}(s)\|_{\mathcal{C}^{-\varepsilon}} ds \\
&\qquad\qquad + |b_0|\int_0^T \|\mathbf{z}^{(p)}(s)\|_{\mathcal{C}^{-\varepsilon}} ds\Big\} \\
&\lesssim \|w_0\|_{\mathcal{C}^{-\varepsilon}} + \sum_{p=0}^{2m-1}\Big\{\sum_{j=1}^{p}|b_p|\|w\|_{X_T^{-\varepsilon,\sigma}}^j \|\mathbf{z}^{(p-j)}\|_{L_T^q\mathcal{C}^{-\varepsilon}}\Big(\int_0^T s^{-\left(\frac{\sigma+\varepsilon}{2}\right)jq'} ds\Big)^{\frac{1}{q'}} \\
&\qquad\qquad + |b_0| T^{\frac{1}{q'}}\|\mathbf{z}^{(p)}\|_{L_T^q\mathcal{C}^{-\varepsilon}}\Big\} \\
&\le Cr + C'(1+r+R)^{2m-1}T^{\frac{1}{q'}-\frac{\sigma+\varepsilon}{2}(2m-1)},
\end{aligned}
$$

for some constants $C, C' > 0$ independant of r, R, and T. Thus, taking $T \sim \left(\frac{r}{(1+r+R)^{2m-1}}\right)^{\theta}$ with $\theta = \left(\frac{1}{q'} - \frac{\sigma+\varepsilon}{2}(2m-1)\right)^{-1} > 0$, we get

$$
\left\|\Phi_{(w_0,\mathbf{z})}(w)\right\|_{L_T^\infty \mathcal{C}^{-\varepsilon}} \le 2Cr.
$$

Similarly as above,

$$
\begin{aligned}
&\sup_{0<t\le T} t^{\frac{\sigma+\varepsilon}{2}}\left\|\Phi_{(w_0,\mathbf{z})}(w)(t)\right\|_{\mathcal{C}^{\sigma}} \\
&\lesssim \|w_0\|_{\mathcal{C}^{-\varepsilon}} + \sum_{p=0}^{2m-1}\Big\{|b_0|\sup_{0<t\le T} t^{\frac{\sigma+\varepsilon}{2}}\int_0^t (t-s)^{-\frac{\sigma+\varepsilon}{2}}\|\mathbf{z}^{(p)}(s)\|_{\mathcal{C}^{-\varepsilon}} ds \\
&\qquad + \sum_{j=0}^{p}|b_p|\sup_{0<t\le T} t^{\frac{\sigma+\varepsilon}{2}}\int_0^t (t-s)^{-\frac{\sigma+\varepsilon}{2}}\|w^j(s)\mathbf{z}^{(p-j)}(s)\|_{\mathcal{C}^{-\varepsilon}} ds\Big\} \\
&\lesssim \|w_0\|_{\mathcal{C}^{-\varepsilon}} + \sum_{p=0}^{2m-1}\Big\{|b_0|\|\mathbf{z}^{(p)}\|_{L_T^q\mathcal{C}^{-\varepsilon}}\sup_{0<t\le T} t^{\frac{\sigma+\varepsilon}{2}}\Big(\int_0^t (t-s)^{-\frac{\sigma+\varepsilon}{2}q'} ds\Big)^{\frac{1}{q'}} \\
&\qquad + \sum_{j=1}^{p}|b_p|\|w\|_{X_T^{-\varepsilon,\sigma}}^j\|\mathbf{z}^{(p-j)}\|_{L_T^q\mathcal{C}^{-\varepsilon}}\sup_{0<t\le T} t^{\frac{\sigma+\varepsilon}{2}}\Big(\int_0^t (t-s)^{-\frac{\sigma+\varepsilon}{2}q'} s^{-\left(\frac{\sigma+\varepsilon}{2}\right)jq'} ds\Big)^{\frac{1}{q'}}\Big\} \\
&\le Cr + C'(1+r+R)^{2m-1}T^{\frac{1}{q'}-\frac{\sigma+\varepsilon}{2}(2m-1)},
\end{aligned}
$$

where we used the elementary integral inequality

$$
\int_0^t (t-s)^{-a}s^{-b} ds \lesssim t^{1-a-b}
$$

for any $0 < a, b < 1$ with $a + b > 1$. Putting the two estimates together with the choice of T yields

$$
\left\|\Phi_{(w_0,\mathbf{z})}(w)\right\|_{X_T^{-\varepsilon,\sigma}} \le 4Cr.
$$

As for the time continuity, take $0 < t \leq T$ and h small enough, then split the integral

$$\begin{aligned}\Phi_{(w_0,\mathbf{z})}(w)(t+h) - \Phi_{(w_0,\mathbf{z})}(w)(t) &= e^{-t\mathcal{H}}(e^{-h\mathcal{H}}-1)w_0 + \int_0^t e^{-(t-s)\mathcal{H}}(e^{-h\mathcal{H}}-1)\mathbf{f}(w,\mathbf{z})(s)\,ds \\ &\quad + \int_t^{t+h} e^{-(t+h-s)\mathcal{H}}\mathbf{f}(w,\mathbf{z})(s)\,ds \\ &=: (1) + (2) + (3).\end{aligned}$$

We use both the Schauder estimate (I.6) and the continuity of the semigroup given by Lemma I.8 to investigate the convergence $h \to 0$. This is straightforward for (1), and for (2) we get

$$\|(2)\|_{\mathcal{C}^\sigma} \lesssim \int_0^t (t-s)^{-\frac{\sigma+\varepsilon}{2}} \|(e^{-h\mathcal{H}}-1)\mathbf{f}(w,\mathbf{z})(s)\|_{\mathcal{C}^{-\varepsilon}}\,ds \underset{h\to 0}{\longrightarrow} 0$$

in view of previous computations and the dominated convergence theorem. Same goes for (3), changing variables and proceeding as above we get

$$\begin{aligned}\|(3)\|_{\mathcal{C}^\sigma} &\lesssim \int_t^{t+h} (t+h-s)^{-\frac{\sigma+\varepsilon}{2}} \|\mathbf{f}(w,\mathbf{z})(s)\|_{\mathcal{C}^{-\varepsilon}}\,ds \\ &\lesssim \int_0^h \tau^{-\frac{\sigma+\varepsilon}{2}} \|\mathbf{f}(w,\mathbf{z})(t+h-\tau)\|_{\mathcal{C}^{-\varepsilon}}\,ds \\ &\lesssim \left(\int_0^h \tau^{-q'\frac{\sigma+\varepsilon}{2}}(t+h-s)^{-q'(2m-1)\frac{\sigma+\varepsilon}{2}}\,ds\right)^{\frac{1}{q'}} \|w\|^{2m-1}_{X_T^{-\varepsilon,\sigma}} \|\mathbf{z}\|_{L^q_{T_0}\mathcal{C}^{-\varepsilon}} \\ &\lesssim t^{-(2m-1)\frac{\sigma+\varepsilon}{2}} h^{\frac{1}{q'}-\frac{\sigma+\varepsilon}{2}} \|w\|^{2m-1}_{X_T^{-\varepsilon,\sigma}} \|\mathbf{z}\|_{L^q_{T_0}\mathcal{C}^{-\varepsilon}} \\ &\underset{h\to 0}{\longrightarrow} 0.\end{aligned}$$

This proves continuity of $\Phi(w)$ as a $\mathcal{C}^\sigma(\mathcal{M})$-valued map on $(0,T]$. Continuity in $\mathcal{C}^{-\varepsilon}(\mathcal{M})$ on $[0,T]$ is obtained via the same arguments, without the blow-up at $t=0$. Altogether, this shows that $\Phi_{(w_0,\mathbf{z})}$ maps the ball of radius $4Cr$ of $X_T^{-\varepsilon,\sigma}$ to itself. Now for w_1, w_2 in this ball, proceeding as above we also have

$$\begin{aligned}&\left\|\Phi_{(w_0,\mathbf{z})}(w_1) - \Phi_{(w_0,\mathbf{z})}(w_2)\right\|_{L^\infty_T\mathcal{C}^{-\varepsilon}} \\ &\lesssim \int_0^T \left\|\mathbf{f}(w_1,\mathbf{z})(s) - \mathbf{f}(w_2,\mathbf{z})(s)\right\|_{\mathcal{C}^{-\varepsilon}} ds \\ &\lesssim \sum_{p=1}^{2m-1}\sum_{j=1}^{p} |b_p| \int_0^T \left\|(w_1^j - w_2^j)\mathbf{z}^{(p-j)}\right\|_{\mathcal{C}^{-\varepsilon}} ds \\ &\lesssim \sum_{p=1}^{2m-1}\sum_{j=1}^{p} |b_p| \int_0^T \|w_1(s)-w_2(s)\|_{\mathcal{C}^\sigma}\left(\|w_1(s)\|^{j-1}_{\mathcal{C}^\sigma} + \|w_2(s)\|^{j-1}_{\mathcal{C}^\sigma}\right)\|\mathbf{z}^{(p-j)}(s)\|_{\mathcal{C}^{-\varepsilon}} ds \\ &\leq C'(1+r+R)^{2m-2} T^{\frac{1}{q'}-\frac{\sigma+\varepsilon}{2}(2m-1)} \|w_1-w_2\|_{X_T^{-\varepsilon,\sigma}},\end{aligned}$$

and

$$\sup_{0<t\leq T} t^{\frac{\sigma+\varepsilon}{2}}\left\|\Phi_{(w_0,\mathbf{z})}(w_1)(t)-\Phi_{(w_0,\mathbf{z})}(w_2)(t)\right\|_{\mathcal{C}^\sigma}$$
$$\leq C'(1+r+R)^{2m-2}T^{\frac{1}{q'}-\frac{\sigma+\varepsilon}{2}(2m-1)}\|w_1-w_2\|_{X_T^{-\varepsilon,\sigma}}.$$

With our choice of T this shows that $\Phi_{(w_0,\mathbf{z})}$ is also a contraction on this ball, thus admits a unique fixed point w which is then a mild solution to (III.27) in $X_T^{-\varepsilon,\sigma}$ which satisfies (III.28). Similar estimates yield for solutions $w,\widetilde{w}$ with data $(w_0,\mathbf{z}),(\widetilde{w_0},\widetilde{\mathbf{z}})$:

$$\begin{aligned}\|w-\widetilde{w}\|_{X_T^{-\varepsilon,\sigma}} &= \left\|\Phi_{(w_0,\mathbf{z})}(w)-\Phi_{(\widetilde{w_0},\widetilde{\mathbf{z}})}(\widetilde{w})\right\|_{X_T^{-\varepsilon,\sigma}}\\
&\leq C\|w_0-\widetilde{w_0}\|_{\mathcal{C}^{-\varepsilon}}\\
&\qquad + C'(1+r+R)^{2m-2}T^{\frac{1}{q'}-\frac{\sigma+\varepsilon}{2}(2m-1)}\|w-\widetilde{w}\|_{X_T^{-\varepsilon,\sigma}}\\
&\qquad + C'(1+r)^{2m-2}T^{\frac{1}{q'}-\frac{\sigma+\varepsilon}{2}(2m-1)}\|\mathbf{z}-\widetilde{\mathbf{z}}\|_{L_T^q\mathcal{C}^{-\varepsilon}}.\end{aligned}$$

This shows the Lipschitz dependence on $(w_0,\mathbf{z})$. ▷

With the local well-posedness result of Proposition III.13, we can prove our main local well-posedness result of Theorem III.2.

Proof of Theorem III.2 – Let ε,σ,q be as in the statement of Theorem III.2. Then for $T_0,R>0$, $0\leq k\leq 2m-1$ and $N\geq 1$, define the events

$$\Sigma_R^{N,k} := \Big\{\omega\in\Omega, \text{such that } \Psi\in C([0;T_0];\mathcal{C}^{-\varepsilon}(\mathcal{M})),\ \Psi^{\diamond k}\in L^q([0;T_0];\mathcal{C}^{-\varepsilon}(\mathcal{M})),$$
$$\|(\mathbf{P}_N\Psi)^{\diamond k}-\Psi^{\diamond k}\|_{L^q_{T_0}\mathcal{C}^{-\varepsilon}}\leq N^{-\frac{\kappa}{2}}R,\ \|(\mathbf{P}_N\Psi)^{\diamond k}\|_{L^q_{T_0}\mathcal{C}^{-\varepsilon}}\leq R\Big\},$$

where κ is as in Proposition III.7. Then set

$$\Sigma_R := \bigcap_{N\in\mathbb{N}^*}\bigcap_{0\leq k\leq 2m-1}\Sigma_R^{N,k}$$

and

$$\Sigma := \liminf_{R\to+\infty}\Sigma_R.$$

From the bounds (III.13) and (III.14) we get

$$\mathbb{P}\left(\Omega\setminus\Sigma_R\right)\leq Ce^{-cR^{\frac{1}{m}}}$$

for some constants $c,C>0$. Then Borel-Cantelli lemma gives that Σ is of full probability.

Given $R>0$, we get from Proposition III.13 that there exists a time $T=T(\|u_0\|_{\mathcal{C}^{-\varepsilon}},R)\in(0,T_0\wedge 1]$ such that for any $\omega\in\Sigma_R$ and $N\geq 1$, (III.27) admits a unique solution $w_N\in X_T^{-\varepsilon,\sigma}$ with data $w_N(0)=u_0-\Psi(0)$. We then obtain a unique solution u_N to (III.9) by setting

$$u_N = \Psi + w_N \in \Psi + X_T^{-\sigma,\varepsilon}.$$

Moreover, since $(\mathbf{P}_N\Psi)^{\diamond k} \to \Psi^{\diamond k}$ in $L^q([0;T_0];\mathcal{C}^{-\varepsilon}(\mathcal{M}))$, the Lipschitz continuity property of the solution in Proposition III.13 ensures that w_N converges in $X_T^{-\varepsilon,\sigma}$ to the solution w to (III.27) with data $(u_0 - \Psi(0), \mathbf{z})$. Since $u_N = \Psi + w_N$, this shows that u_N converges in $\Psi + X_T^{-\varepsilon,\sigma}$ to $u = \Psi + w$. $\triangleright$

III.3.3 Almost sure globalization and measure invariance

We now turn to the globalization problem of the truncated dynamics (III.9) for random initial data u_0 with law given by ρ_N defined as in (III.8). Note that since Theorem III.2 holds $\mathbb{P}$ almost surely for any initial data in $\mathcal{C}^{-\varepsilon}(\mathcal{M})$, in view of Proposition III.7, it then also holds $\rho \otimes \mathbb{P}$ almost surely. Moreover, recall that Proposition III.13 provides a blow-up alternative: since the local time of existence $T = T(\|u_0\|_{\mathcal{C}^{-\varepsilon}}, \|\mathbf{z}\|_{L^q_{T_0}\mathcal{C}^{-\varepsilon}})$ only depends on $\|u_0\|_{\mathcal{C}^{-\varepsilon}}$ and $\|\mathbf{z}\|_{L^q_{T_0}\mathcal{C}^{-\varepsilon}}$, for any $T_0 > 0$, if $T^* = T^*(u_0,\omega) > 0$ is the maximal time of existence of w_N on $[0,T_0]$, then

$$\sup_{0\le t<T^*} \|w_N(t)\|_{\mathcal{C}^{-\varepsilon}} = +\infty \qquad \text{or} \qquad T^* = T_0. \tag{III.29}$$

As we intend to prove global well-posedness for random initial data, we first need to investigate measure invariance of ρ_N under the flow of the truncated equation (III.9). As $\mathbf{P}_N$ does not make the truncated equation (III.9) a finite-dimensional SDE, we first study another truncated dynamics. Let $M > 0$ and set Π_M to be the orthogonal projection of the finite dimensional space Span $\{\varphi_n\,,\ \lambda_n \le M\}$, we denote by d_M its dimension, with $d_M \simeq M$ according to Weyl's law (I.3). Π_M being a sharp projection, we do not recover continuity in Hölder spaces, so we rather consider a smooth counterpart $\chi_M = \chi(M^{-1}\mathcal{H})$ for some $\chi \in C_0^\infty(\mathbb{R})$ compactly supported in $(-1;1)$. Note that $\mathbf{P}_N$ is based on Δ while χ_M is defined with $\mathcal{H}$, hence both operators do not commute with one another. Still, we consider $u_{N,M}$ to solve the following equation starting from some initial data u_0:

$$\begin{cases} \partial_t u_{N,M} + \mathcal{H}u_{N,M} + \chi_M\mathbf{P}_N f_N^\diamond\left(\mathbf{P}_N\chi_M u_{N,M}\right) = \sqrt{2}\zeta, \\ u_{N,M}(0) = u_0. \end{cases} \tag{III.30}$$

Expanding $f_N^\diamond$ to emphasize the dependency in N, we can rewrite (III.30) as

$$\begin{cases} \partial_t u_{N,M} + \mathcal{H}u_{N,M} + \sum_{p=0}^{2m-1} b_p\chi_M\mathbf{P}_N H_p\left(\mathbf{P}_N\chi_M u_{N,M},\sigma_N\right) = \sqrt{2}\zeta, \\ u_{N,M}(0) = u_0. \end{cases} \tag{III.31}$$

Note that χ_M acts on both the non-linearity and the $u_{N,M}$ term inside, and since $\chi_M = \Pi_M\chi_M = \chi_M\Pi_M$ by the support property of χ, we see that $\Pi_M u_{N,M}$ will solve a finite dimensional system of coupled SDEs while $(1-\Pi_M)u_{N,M}$ satisfies a linear evolution equation driven by a white noise. Namely, if we write $\Pi_M u_{N,M}(t) = \sum_{n=0}^{d_M} a_{N,M}^{(n)}(t)\varphi_n$, then for $0 \leq n \leq d_M$ we have

$$da^{(n)}_{N,M} + \left[\lambda_n a^{(n)}_{N,M} + \sum_{p=0}^{2m-1} b_p \left\langle \chi_M \mathbf{P}_N H_p \left(\mathbf{P}_N \chi_M \sum_{n=0}^{d_M} a^{(n)}_{N,M}\varphi_n, \sigma_N \right), \varphi_n \right\rangle \right] dt = \sqrt{2}\, dB_n$$

for independent Brownian motions B_n. This can be seen as a finite dimensional stochastic gradient flow

$$da^{(n)} + \partial_{a^{(n)}} \mathcal{E}_{N,M}(a^{(0)}, \cdots, a^{(d_M)})\, dt = \sqrt{2}\, dB_n \quad \text{for } 0 \leq n \leq d_M$$

with truncated energy

$$\mathcal{E}_{N,M}(a^{(0)}, \cdots, a^{(d_M)}) := \frac{1}{2}\sum_{n=0}^{d_M} \lambda_n (a^{(n)})^2 + \sum_{p=0}^{2m-1} \frac{b_p}{p+1} \int_{\mathcal{M}} H_{p+1}\left(\mathbf{P}_N \chi_M \sum_{n=0}^{d_M} a^{(n)}_{N,M}\varphi_n, \sigma_N \right) dx.$$

To this finite dimensional system correspond the generator $\mathcal{L}_{N,M}$ and Gibbs measure $\rho_{N,M}$ defined respectively by

$$\mathcal{L}_{N,M} := \sum_{n=0}^{d_M} \partial^2_{a^{(n)}} - \partial_{a^{(n)}} \mathcal{E}_{N,M}(a^{(0)}, \cdots, a^{(d_M)}) \partial_{a^{(n)}}$$

and

$$\begin{aligned} d\rho_{N,M}(a^{(0)}, \cdots, a^{(d_M)}) &:= \mathcal{Z}^{-1}_{N,M} e^{-\mathcal{E}_{N,M}(a^{(0)}, \cdots, a^{(d_M)})}\, da^{(0)} \cdots da^{(d_M)} \\ &= \mathcal{Z}^{-1}_{N,M} R_{N,M}(a^{(0)}, \cdots, a^{(d_M)}) d((\Pi_M)_* \mu^{\mathcal{H}})(a^{(0)}, \cdots, a^{(d_M)}), \end{aligned}$$

with normalization constant $\mathcal{Z}_{N,M}$. Integrating $\mathcal{L}_{N,M} f$ against $d\rho_{N,M}$ for some test function f we get (denoting $a^{(0-d_M)} = (a^{(0)}, \cdots, a^{(d_M)})$)

$$\begin{aligned} \int_{\mathbb{R}^M} \mathcal{L}_{N,M} f(a^{(0-d_M)})\, d\rho_{N,M}(a^{(0-d_M)}) &= \mathcal{Z}^{-1}_{N,M} \int_{\mathbb{R}^M} e^{-\mathcal{E}_{N,M}(a^{(0-d_M)})} \sum_{n=0}^{d_M} \Big(\partial^2_{a^{(n)}} f(a^{(0-d_M)}) \\ &\qquad - \partial_{a^{(n)}} \mathcal{E}_{N,M} \partial_{a^{(n)}} f(a^{(0-d_M)}) \Big)\, da^{(0)} \cdots da^{(d_M)} \\ &= 0 \end{aligned}$$

after integration by parts. This proves that $\Pi_M u_{N,M}$ leaves the measure $\rho_{N,M}$ invariant. Now for remaining part $u^{\perp}_{N,M} := (1 - \Pi_M) u_{N,M}$, the non-linear term disappears since $(1 - \Pi_M)\chi_M = 0$ by the support property of χ, so that $u^{\perp}_{N,M}$ satisfies the linear equation

$$\partial_t u^{\perp}_{N,M} + \mathcal{H} u^{\perp}_{N,M} = \sqrt{2}(1 - \Pi_M)\zeta.$$

This equation has as unique invariant measure $(1 - \Pi_M)_* \mu^{\mathcal{H}}$. Therefore, $u_{N,M} = \Pi_M u_{N,M} + u^{\perp}_{N,M}$ leaves the measure $\rho_{N,M} \otimes (1 - \Pi_M)_* \mu^{\mathcal{H}}$ invariant. The only part missing to make this whole discussion rigorous is the global existence of a solution $u_{N,M}$ in $\mathcal{C}^{-\varepsilon}(\mathcal{M})$, hence the following proposition.

Proposition III.14 – *For fixed $N, M \in \mathbb{N}^*$, and initial data $u_0 \in \mathcal{C}^{-\varepsilon}$, then the flow of (III.31) extends globally. Moreover the truncated measure $\rho_{N,M} \otimes (1-\Pi_M)^* \mu^{\mathcal{H}}$ is invariant under the flow of (III.31).*

Proof – First we prove global well-posedness. As per usual set $u_{N,M} = \overset{\bullet}{|} + w_{N,M}$ where $\overset{\bullet}{|}$ is the stochastic convolution as before and $w_{N,M}$ solves the non-linear equation

$$
\begin{cases}
\partial_t w_{N,M} + \mathcal{H} w_{N,M} + \sum_{p=0}^{2m-1} b_p \chi_M \mathbf{P}_N H_p\left(\mathbf{P}_N \chi_M w_{N,M} + \mathbf{P}_N \chi_M \overset{\bullet}{|}, \sigma_N\right) = 0, \\
w_{N,M}(0) = u_0 - \overset{\bullet}{|}(0).
\end{cases}
\tag{III.32}
$$

Then, due to Proposition III.8, local existence of $w_{N,M}$ falls under the scope of Proposition III.13, and we only need to prove that solutions exist globally in time. According to the blow-up criterion above, we only have to make sure that the $\mathcal{C}^{-\varepsilon}(\mathcal{M})$ norm of $w_{N,M}$ does not blow-up in finite time. Note that applying Π_M to (III.32), we see that $\Pi_M w_{N,M}$ solves (III.32) with initial data $\Pi_M\big(u_0 - \overset{\bullet}{|}(0)\big)$ while $(1-\Pi_M)w_{N,M}$ solves a linear equation, thus exists globally. Note also that we have by Sobolev inequality

$$
\|\Pi_M w_{N,M}(t)\|_{\mathcal{C}^{-\varepsilon}} \lesssim \|\Pi_M w_{N,M}(t)\|_{H^{1-\frac{\varepsilon}{2}}} \sim \|\Pi_M w_{N,M}(t)\|_{\mathcal{D}^{1-\frac{\varepsilon}{2}}} \lesssim_M \|\Pi_M w_{N,M}(t)\|_{L^2}.
$$

Thus we only need to bound the $L^2(\mathcal{M})$ norm of $\Pi_M w_{N,M}$ to get global existence for each fixed N, M. This is done by estimating the following energy

$$
\Psi_{N,M}(t) := \frac{1}{2}\int_{\mathcal{M}} |\Pi_M w_{N,M}(s,x)|^2 \, dx + b_{2m-1}\int_0^t \int_{\mathcal{M}} |\mathbf{P}_N \chi_M \Pi_M w_{N,M}(s,x)|^{2m} \, ds \, dx.
$$

We prove that $\Psi_{N,M}$ is bounded on any interval $[0,T]$ by some finite positive constant, depending on T. Dropping the N, M subscript in $\Pi_M w_{N,M} =: w$ for the sake of clarity and differentiating in t, we get

$$
\begin{aligned}
&\frac{d}{dt}\Psi_{N,M} \\
&= \int_{\mathcal{M}} w\partial_t w + b_{2m-1}\int_{\mathcal{M}} |\mathbf{P}_N \chi_M w|^{2m} \\
&= -\int_{\mathcal{M}} |\mathcal{H}^{\frac{1}{2}} w|^2 - \int_{\mathcal{M}} f_N^{\diamond}(\mathbf{P}_N\chi_M w + \mathbf{P}_N \chi_M \overset{\bullet}{|})\mathbf{P}_N\chi_M w + b_{2m-1}\int_{\mathcal{M}} |\mathbf{P}_N\chi_M w|^{2m} \\
&= -\int_{\mathcal{M}} |\mathcal{H}^{\frac{1}{2}} w|^2 - \sum_{p=0}^{2m-1}\sum_{j=0}^{p} \binom{p}{j} b_p \int_{\mathcal{M}} (\mathbf{P}_N\chi_M w)^{p-j+1}(\mathbf{P}_N\chi_M \overset{\bullet}{|})^{\diamond j} + b_{2m-1}\int_{\mathcal{M}} |\mathbf{P}_N\chi_M w|^{2m} \\
&= -\int_{\mathcal{M}} |\mathcal{H}^{\frac{1}{2}} w|^2 - \sum_{p=0}^{2m-1}\sum_{j=0}^{p} \mathbf{1}_{(j,p)\neq(0,2m-1)} \binom{p}{j} b_p \int_{\mathcal{M}} (\mathbf{P}_N\chi_M w)^{p-j+1}(\mathbf{P}_N\chi_M \overset{\bullet}{|})^{\diamond j}.
\end{aligned}
$$

Having only powers at most $2m-1$ of $\mathbf{P}_N\chi_M w$, we can fix $\delta\in(0,1)$ and use Young's estimate to get

$$\begin{aligned}\left|\int_{\mathcal{M}}(\mathbf{P}_N\chi_M w)^{p-j+1}(\mathbf{P}_N\chi_M \mathring{|})^{\diamond j}\right| &\leq \|\mathbf{P}_N\chi_M w\|_{L^{p-j+1}}^{p-j+1}\|(\mathbf{P}_N\chi_M \mathring{|})^{\diamond j}\|_{L^\infty}\\ &\leq C_{j,p}\delta^{\frac{2m}{p-j+1}}\|\mathbf{P}_N\chi_M w\|_{L^{2m}}^{2m}+C_{j,p}(\delta)\|(\mathbf{P}_N\chi_M \mathring{|})^{\diamond j}\|_{L^\infty}^{\frac{2m}{2m-p+j-1}}\\ &\leq C_{j,p}\delta\|\mathbf{P}_N\chi_M w\mathring{|}\|_{L^{2m}}^{2m}+C_{j,p}(\delta)\|(\mathbf{P}_N\chi_M \mathring{|})^{\diamond j}\|_{L^\infty}^{\frac{2m}{2m-p+j-1}}.\end{aligned}$$

And then after a proper rescale of δ :

$$\begin{aligned}&\left|\sum_{p=0}^{2m-1}\sum_{j=0}^{p}\mathbf{1}_{(j,p)\neq(0,2m-1)}\binom{p}{j}b_p\int_{\mathcal{M}}(\mathbf{P}_N\chi_M w)^{p-j+1}(\mathbf{P}_N\chi_M \mathring{|})^{\diamond j}\right|\\ &\qquad\leq \delta\|\mathbf{P}_N\chi_M w\|_{L^{2m}}^{2m}+C(\delta,T,N,M),\end{aligned}$$

for some constant C. Then, as our non-linearity is defocusing ($b_{2m-1}>0$), integration in time yields

$$\Psi_{N,M}(t)\leq \delta\Psi_{N,M}(t)+C(\delta,T,N,M),$$

and then $\Psi_{N,M}(t)$ is bounded on $[0,T]$. Since $\Psi_{N,M}$ controls the $L^2(\mathcal{M})$ norm of $w=\Pi_M w_{N,M}$, we obtain global existence of $w_{N,M}$, and then global existence of $u_{N,M}$.

Now that we have proved that (III.31) is globally well-posed, the previous discussion ensures that $\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}$ is invariant under the flow of (III.31). ▷

We now turn to the limiting procedure by taking $M,N\to+\infty$. At first we let $M\to+\infty$, this will prove the following bound on the local flow Φ_N of (III.9).

Lemma III.15 – *For any $T>0$ and $N\geq 1$, there exists a constant C_T uniform in $N\in\mathbb{N}^*$ such that*

$$\mathbb{E}_{\rho_N\otimes\mathbb{P}}\Big[\sup_{0\leq t\leq T\wedge T^*}\Big\|\Phi_N(u_0,\omega)(t)\Big\|_{\mathcal{C}^{-\varepsilon}}\Big]\leq C_T,\tag{III.33}$$

where $T^*=T^*_N(u_0,\omega)$ is the maximal time of existence of $u_N=\Phi_N(u_0,\omega)$.

Proof – Let $T>0$ and $N,M\in\mathbb{N}^*$, we first prove a similar bound for $\Phi_{N,M}$ the flow of (III.31). From Proposition III.14, $\Phi_{N,M}$ is globally well-defined by

$$\Phi_{N,M}(u_0,\omega)(t)=\mathring{|}(\omega)(t)+w_{N,M}(t)$$

where we see $w_{N,M}(t)$ as a continuous function of the stochastic data $(\mathring{|}^{\diamond p})_{0\leq p\leq 2m-1}$, as showed in Proposition III.13. Thus $\Phi_{N,M}$ satisfies the Duhamel formula for $t\geq 0$

$$\Phi_{N,M}(u_0,\omega)(t)=e^{-t\mathcal{H}}\Big(u_0-\mathring{|}(\omega)(0)\Big)+\mathring{|}(\omega)(t)$$

$$-\int_0^t e^{-(t-s)\mathcal{H}}\chi_M\mathbf{P}_N f_N^\diamond(\mathbf{P}_N\chi_M\Phi_{N,M}(u_0,\omega)(s))\,ds.$$

Hence taking the $\mathcal{C}^{-\varepsilon}(\mathcal{M})$ norm and then the supremum in t yields

$$\begin{aligned}\sup_{0\le t\le T}\|\Phi_{N,M}(u_0,\omega)(t)\|_{\mathcal{C}^{-\varepsilon}} &\lesssim \sup_{0\le t\le T}\|e^{-t\mathcal{H}}\Big(u_0-\Psi(\omega)(0)\Big)+\Psi(\omega)(t)\|_{\mathcal{C}^{-\varepsilon}}\\ &\quad+\int_0^T\|\chi_M\mathbf{P}_N f_N^\diamond(\mathbf{P}_N\chi_M\Phi_{N,M}(u_0,\omega)(s))\|_{\mathcal{C}^{-\varepsilon}}\,ds\\ &\lesssim \sup_{0\le t\le T}\|e^{-t\mathcal{H}}\Big(u_0-\Psi(\omega)(0)\Big)+\Psi(\omega)(t)\|_{\mathcal{C}^{-\varepsilon}}\\ &\quad+\int_0^T\|f_N^\diamond(\mathbf{P}_N\chi_M\Phi_{N,M}(u_0,\omega)(s))\|_{\mathcal{C}^{-\frac{\varepsilon}{2}}}\,ds,\end{aligned}$$

where we used successively Corollary I.14 and Lemma A.7 for the boundedness properties of χ_M and $\mathbf{P}_N$ on $\mathcal{C}^{-\varepsilon}(\mathcal{M})$. Note that $u_0\sim\rho_N$, so in particular we also have $u_0\in\mathcal{C}^{-\frac{\varepsilon}{2}}$ almost surely, and thus also $\Phi_{N,M}(u_0)(t)\in\mathcal{C}^{-\frac{\varepsilon}{2}}$, so that the last term above makes sense. Integrating with respect to $\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}\otimes\mathbb{P}$, Fubini's theorem and invariance of $\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}$ under $\Phi_{N,M}$ allow to deal with the integral in the right hand side:

$$\begin{aligned}&\mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}\otimes\mathbb{P}}\Big[\int_0^T\|f_N^\diamond(\mathbf{P}_N\chi_M\Phi_{N,M}(u_0,\omega)(s))\|_{\mathcal{C}^{-\frac{\varepsilon}{2}}}\,ds\Big]\\ &\qquad=\int_0^T\mathbb{E}_{\mathbb{P}}\mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}}\Big[\|f_N^\diamond(\mathbf{P}_N\chi_M\Phi_{N,M}(u_0,\omega)(s))\|_{\mathcal{C}^{-\frac{\varepsilon}{2}}}\Big]ds\\ &\qquad=T\mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}}\Big[\|f_N^\diamond(\mathbf{P}_N\chi_M u_0)\|_{\mathcal{C}^{-\frac{\varepsilon}{2}}}\Big]\\ &\qquad=T\mathbb{E}_{\mu^{\mathcal{H}}}\Big[\mathcal{Z}_{N,M}^{-1}R_N(\chi_M u_0)\|f_N^\diamond(\mathbf{P}_N\chi_M u_0)\|_{\mathcal{C}^{-\frac{\varepsilon}{2}}}\Big]\\ &\qquad\le T\Big\|\|f_N^\diamond(\mathbf{P}_N\chi_M u_0)\|_{\mathcal{C}^{-\frac{\varepsilon}{2}}}\Big\|_{L^2(\mu^{\mathcal{H}})}\Big\|\mathcal{Z}_{N,M}^{-1}R_N(\chi_M u_0)\Big\|_{L^2(\mu^{\mathcal{H}})}.\end{aligned}$$

From there, we invoke Propositions III.8 and III.12 to get a bound uniform in both N and M.

For the other part, note that $e^{-t\mathcal{H}}\Big(u_0-\Psi(\omega)(0)\Big)+\Psi(\omega)(t)$ is explicitly given by

$$e^{-t\mathcal{H}}\Big(u_0-\Psi(\omega)(0)\Big)+\Psi(\omega)(t)=e^{-t\mathcal{H}}u_0+\int_0^t e^{-(t-s)\mathcal{H}}\,\zeta(ds).$$

We treat the propagator in a similar way we did for the Duhamel integral

$$\begin{aligned}\mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}\otimes\mathbb{P}}\Big[\sup_{0\le t\le T}\|e^{-t\mathcal{H}}u_0\|_{\mathcal{C}^{-\varepsilon}}\Big]&\lesssim\mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}}\Big[\|u_0\|_{\mathcal{C}^{-\varepsilon}}\Big]\\ &\lesssim\mathbb{E}_{\mu^{\mathcal{H}}}\Big[\|u_0\|^2_{\mathcal{C}^{-\varepsilon}}\Big]^{\frac{1}{2}}\mathbb{E}_{\mu^{\mathcal{H}}}\Big[\mathcal{Z}_{N,M}^{-2}R_N(\chi_M u_0)^2\Big]^{\frac{1}{2}},\end{aligned}$$

and again the right hand side is uniformly bounded in N, M. For the remaining part, we use the Burkholder-Davis-Gundy inequality and finally get

$$\mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}\otimes\mathbb{P}}\Big[\sup_{0\leq t\leq T}\Big\|e^{-t\mathcal{H}}u_0+\int_0^t e^{-(t-s)\mathcal{H}}\zeta(ds)\Big\|_{\mathcal{C}^{-\varepsilon}}\Big]\leq C_T$$

for some constant C_T uniform in N and M. This proves that

$$\mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}\otimes\mathbb{P}}\Big[\sup_{0\leq t\leq T}\|\Phi_{N,M}(u_0,\omega)(t)\|_{\mathcal{C}^{-\varepsilon}}\Big]\leq C_T.$$

Now, from C_T being uniform in N, M, the convergence almost surely and in $L^p(\Omega)$ of $(\mathbf{P}_N\chi_M\Psi)^{\diamond k}$ to $(\mathbf{P}_N\Psi)^{\diamond k}$ in $L^q([0;T];\mathcal{C}^{-\varepsilon}(\mathcal{M}))$ (Proposition III.8) together with the continuity property of the flow given by Proposition III.13, and the convergence in total variation of $\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}$ to ρ_N due to Propositions III.8 and III.12, we can use Fatou's lemma and pass M to the limit so that all in all we have

$$\begin{aligned}&\mathbb{E}_{\rho_N\otimes\mathbb{P}}\Big[\sup_{0\leq t\leq T\wedge T^*}\|\Phi_N(u_0,\omega)(t)\|_{\mathcal{C}^{-\varepsilon}}\Big]\\ &\quad\leq\liminf_{M\to+\infty}\mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}\otimes\mathbb{P}}\Big[\sup_{0\leq t\leq T\wedge T^*}\|\Phi_{N,M}(u_0,\omega)(t)\|_{\mathcal{C}^{-\varepsilon}}\Big]\\ &\quad\leq C_T,\end{aligned}$$

again uniformly in $N\in\mathbb{N}^*$. ▷

In view of the blow-up criterion (III.29) for the equation (III.22), thus for the equation (III.9), the estimate (III.33) is enough to prove probabilistic global well-posedness of the truncated renormalized dynamics (III.9).

Proposition III.16 – *For any $N\geq 1$, (III.9) is $\rho_N\otimes\mathbb{P}$-almost surely globally well posed. Moreover the truncated Gibbs measure ρ_N is invariant under the flow Φ_N of (III.9), that is, for any $t\geq 0$ and $\mathfrak{F}\in C_b(\mathcal{C}^{-\varepsilon}(\mathcal{M}))$,*

$$\mathbb{E}_{\rho_N\otimes\mathbb{P}}\Big[\mathfrak{F}\left[\Phi_N(u_0,\omega)(t)\right]\Big]=\mathbb{E}_{\rho_N}\Big[\mathfrak{F}\left[u_0\right]\Big].$$

Proof – Fix $N\geq 1$, then (III.33) reads as, uniformly in N, for each T there exists an event $\Sigma_{N,T}$ of full $\rho_N^{\mathcal{H}}\otimes\mathbb{P}$-measure such that

$$\|\Phi_N(u_0,\omega)(t)\|_{\mathcal{C}^{-\varepsilon}}\leq C_T$$

for any $(u_0,\omega)\in\Sigma_{N,T}$. Specializing to times $k\in\mathbb{N}^*$, then the blow-up alternative ensures that $\Phi_N(u_0,\omega)$ is well defined at least up to time k. Taking the countable intersection, we see that

$$\Sigma_N:=\bigcap_{k\in\mathbb{N}^*}\Sigma_{N,T_k}$$

is of full $\rho_N\otimes\mathbb{P}$-measure and that Φ_N is defined globally for any $(u_0,\omega)\in\Sigma_N$.
Now for the invariance part, pick $t\geq 0$ and $\mathfrak{F}\in C_b(\mathcal{C}^{-\varepsilon}(\mathcal{M}))$. As in Proposi-

tion III.10, we write ρ_N as

$$d\rho_N(u) = \mathcal{Z}_N^{-1} R_N(u)\, d\mu^{\mathcal{H}}(u).$$

Then from the convergence in total variation of $\rho_{N,M} \otimes (1-\Pi_M)^* \mu^{\mathcal{H}}$ to ρ_N and that of $\Phi_{N,M}$ to Φ_N, we have

$$\mathbb{E}_{\rho_N \otimes \mathbb{P}}\Big[\mathfrak{F}\left[\Phi_N(u_0,\omega)(t)\right]\Big] = \lim_{M\to+\infty} \mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}\otimes\mathbb{P}}\Big[\mathfrak{F}\left[\Phi_{N,M}(u_0,\omega)(t)\right]\Big],$$

and from Proposition III.14 we can deduce that

$$\begin{aligned}\mathbb{E}_{\rho_N \otimes \mathbb{P}}\Big[\mathfrak{F}\left[\Phi_N(u_0,\omega)(t)\right]\Big] &= \mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}\otimes\mathbb{P}}\Big[\mathfrak{F}\left[\Phi_{N,M}(u_0,\omega)(t)\right]\Big] \\ &= \lim_{M\to+\infty} \mathbb{E}_{\rho_{N,M}\otimes(1-\Pi_M)^*\mu^{\mathcal{H}}}\Big[\mathfrak{F}\left[u_0\right]\Big] = \mathbb{E}_{\rho_N}\Big[\mathfrak{F}\left[u_0\right]\Big],\end{aligned}$$

proving that ρ_N is invariant under the flow Φ_N. ▷

Combining now the a priori estimate of Proposition (III.33) together with the local convergence of Φ_N to Φ (Theorem III.2) and that of ρ_N to ρ (Proposition III.7), we can finally get our almost sure global well-posedness result.

Proof of Theorem III.3 – Recall that the constant in (III.33) is uniform in N. Since Propositions III.13 and III.1 ensure convergence of Φ_N and ρ_N respectively almost surely and in total variation, Fatou's lemma yet again yields

$$\begin{aligned}\mathbb{E}_{\rho\otimes\mathbb{P}}\Big[\sup_{0\le t\le T\wedge T^*} \|\Phi(u_0,\omega)(t)\|_{\mathcal{C}^{-\varepsilon}}\Big] &\le \liminf_{N\to+\infty} \mathbb{E}_{\rho_N\otimes\mathbb{P}}\Big[\sup_{0\le t\le T\wedge T^*} \|\Phi_N(u_0,\omega)(t)\|_{\mathcal{C}^{-\varepsilon}}\Big] \\ &\le C_T,\end{aligned}$$

where $T^* = T^*(u_0,\omega)$ is the maximal time of existence of $u = \Phi(u_0,\omega)$. As before, this yields the existence of a set [4] $\tilde{\Sigma} \subset \mathcal{C}^{-\varepsilon}(\mathcal{M}) \times \Sigma$ of full $\rho\otimes\mathbb{P}$-measure such that $\Phi(u_0,\omega)$ is defined globally for any $(u_0,\omega) \in \tilde{\Sigma}$. The invariance of ρ then follows again from dominated convergence, the convergence of Φ_N to Φ (Theorem III.2), the convergence of ρ_N to ρ (Theorem III.1), and the invariance of ρ_N under the flow Φ_N (Proposition III.16): for any $t \ge 0$ and $\mathfrak{F} \in C_b(\mathcal{C}^{-\varepsilon}(\mathcal{M}))$ we get

$$\begin{aligned}\mathbb{E}_{\rho\otimes\mathbb{P}}\Big[\mathfrak{F}\left[\Phi(u_0,\omega)(t)\right]\Big] &= \lim_{N\to+\infty} \mathbb{E}_{\rho_N\otimes\mathbb{P}}\Big[\mathfrak{F}\left[\Phi_N(u_0,\omega)(t)\right]\Big] \\ &= \lim_{N\to+\infty} \mathbb{E}_{\rho_N}\Big[\mathfrak{F}\left[u_0\right]\Big] = \mathbb{E}_{\rho}\Big[\mathfrak{F}\left[u_0\right]\Big]\end{aligned}$$

This concludes the proof of Theorem III.3. ▷

4. Recall that $\Sigma \subset \Omega$ is the set defined in Theorem III.2 where almost sure local well-posedness of (III.10) holds.

IV Ergodicity for the Anderson Φ^4_2 measure

Contents

IV.1 – A first glance at the equation

We consider this time the Anderson Φ^4_2 equation posed on the 2 dimensional torus $\mathbb{T}^2$

$$\begin{cases} \partial_t u + \mathcal{H}u + u^{\diamond 3} = \sqrt{2}\zeta, \\ u_{t=0} = u_0. \end{cases} \tag{IV.1}$$

Note that in this Chapter, we will write the time dependency of functions as a subscript, e.g. $u_t = u(t,\cdot)$ to lighten the notations.
As far as solving the equation is concerned, local well-posedness has already been dealt with in Chapter III. This time instead, we would rather need the process Ψ from before to start from 0. Define for $s \in \mathbb{R}$ the solution to the linear equation starting from 0 at time s

$$\begin{cases} \partial_t \Psi_s + \mathcal{H}\Psi_s = \sqrt{2}\zeta, \\ \Psi_{s,s} = 0. \end{cases} \tag{IV.2}$$

For $s = -\infty$, we recover the stationary process $\Psi_{-\infty,\cdot} = \Psi$ defined in (III.12), note that Ψ_s can then be explicitely written in terms of $\Psi_{-\infty}$

$$\begin{aligned}\Psi_{s,t} &= \int_s^t e^{-(t-\tau)\mathcal{H}}\zeta(d\tau) \\ &= \int_{-\infty}^t e^{-(t-\tau)\mathcal{H}}\zeta(d\tau) - e^{-(t-s)\mathcal{H}}\int_{-\infty}^s e^{-(t-\tau)\mathcal{H}}\zeta(d\tau) \\ &= \Psi_{-\infty,t} - e^{-(t-s)\mathcal{H}}\Psi_{-\infty,s}\end{aligned}$$

Let also $\mathcal{F}_s$ be the white noise filtration, that is the natural augmentation [1] of the filtration generated by

$$\left\{\langle\zeta,\psi\rangle\,,\ \psi\in L^2(\mathbb{R}\times\mathbb{T}^2)\text{ such that }\psi_{|(s,+\infty)\times\mathbb{T}^2}=0\right\}.$$

First thing to note is that, as ζ is white in time, $\Psi_{s,t}$ is independent of $\mathcal{F}_s$ for $s < t$ (increments of ζ between s and t are independent of $\mathcal{F}_s$). As for fixed $s < t$, $e^{-(t-s)\mathcal{H}}\Psi_{-\infty,s}$ is almost $\mathcal{C}^1$ in space, the following binomial expansion for defining the k-th power of Ψ_s is consistent

$$\Psi^{k}_{s,t} := \sum_{j=0}^k (-1)^j\binom{k}{j}\left(e^{-(t-s)\mathcal{H}}\Psi_{-\infty,s}\right)^j\Psi^{\diamond(k-j)}_{-\infty,t}. \tag{IV.3}$$

In fact, one can carry out the exact same renormalization procedure as we did directly on $\Psi_{s,\cdot}$ to define the powers $\Psi^{k}_{s,\cdot}$. For $N\in\mathbb{N}^*$, let σ_N be as in Equation (III.6), applying the regularizing operator $\mathbf{P}_N$ and using the binomial formula Lemma III.5 for Hermite polynomials we get

$$\begin{aligned}\Psi^{k\,(N)}_{s,t} &:= H_k\left(\mathbf{P}_N\Psi_{s,t},\sigma_N^2\right) = \sum_{j=0}^k(-1)^j\binom{k}{j}\left(\mathbf{P}_N\left(e^{-(t-s)\mathcal{H}}\Psi_{-\infty,s}\right)\right)^j H_{k-j}\left(\mathbf{P}_N\Psi_{-\infty,t},\sigma_N^2\right) \\ &= \sum_{j=0}^k(-1)^j\binom{k}{j}\left(\mathbf{P}_N\left(e^{-(t-s)\mathcal{H}}\Psi_{-\infty,s}\right)\right)^j\left(\mathbf{P}_N\Psi_{-\infty,t}\right)^{\diamond(k-j)}.\end{aligned}$$

In view of Proposition III.7, $\Psi^{k}_{s,\cdot}$ inherits the same regularity properties as $\Psi^{\diamond k}_{-\infty}$, we can prove the following lemma.

Lemma IV.1 – *For $k\in\{1,2,3\}$, and $-\infty < s$, for any $\varepsilon\in(0,1)$, $T>0$ and $p\geq q\geq 1$*

$$\Psi^{k\,(N)}_{s,s+\cdot} \underset{N\to+\infty}{\to} \Psi^{k}_{s,s+\cdot}\ \textit{ in } L^p\left(\Omega,L^q([0,T],\mathcal{C}^{-\varepsilon})\right).$$

1. see for instance [RY04], Chapter I.4

Moreover $\overset{k}{\vee}_{s,s+\cdot}$ *is independent of the white noise filtration* $\mathcal{F}_s$ *and for any* $-\infty < s_1, s_2$*, the processes* $\overset{k}{\vee}_{s_1,s_1+\cdot}$ *and* $\overset{k}{\vee}_{s_2,s_2+\cdot}$ *have the same law.*

Proof – From the definition of $\overset{k}{\vee}^{(N)}$, we have for $h > 0$

$$\overset{k}{\vee}^{(N)}_{s,s+h} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \Big(\mathbf{P}_N\big(e^{-h\mathcal{H}}\Psi_{-\infty,s}\big)\Big)^j \Big(\mathbf{P}_N \Psi_{-\infty,s+h}\Big)^{\diamond(k-j)}.$$

The convergence then follows from Proposition III.7 by splitting the product

$$\begin{aligned}
&\Big(\mathbf{P}_N\big(e^{-h\mathcal{H}}\Psi_{-\infty,s}\big)\Big)^j \Big(\mathbf{P}_N \Psi_{-\infty,s+h}\Big)^{\diamond(k-j)} - \Big(\big(e^{-h\mathcal{H}}\Psi_{-\infty,s}\big)\Big)^j \Big(\Psi_{-\infty,s+h}\Big)^{\diamond(k-j)} \\
&\qquad = \Big(\mathbf{P}_N\big(e^{-h\mathcal{H}}\Psi_{-\infty,s}\big)\Big)^j \Big[\Big(\mathbf{P}_N \Psi_{-\infty,s+h}\Big)^{\diamond(k-j)} - \Psi^{\diamond(k-j)}_{-\infty,s+h}\Big] \\
&\qquad\quad + \Big[\Big(\mathbf{P}_N\big(e^{-h\mathcal{H}}\Psi_{-\infty,s}\big)\Big)^j - \Big(\big(e^{-h\mathcal{H}}\Psi_{-\infty,s}\big)\Big)^j\Big] \Psi^{\diamond(k-j)}_{-\infty,s+h}.
\end{aligned}$$

For the second part of the claim, following the very same steps as in the proof of Proposition III.7, $\mathbf{P}_N \Psi_{s,t}$ is a centered Gaussian variable with covariance kernel given by the regularized Anderson Green's function

$$G_N^{\mathcal{H}}(x,y,t,s) = \sum_{n\geq 0} \frac{\mathbf{P}_N\varphi_n(x)\mathbf{P}_N\varphi_n(y)}{\lambda_n}(1 - e^{-2(t-s)\lambda_n}).$$

Note that the covariance kernel then depends only on the difference $t - s$, thus proving that $\overset{k}{\vee}^{(N)}_{s_1,s_1+\cdot}$ has the same law as $\overset{k}{\vee}^{(N)}_{s_2,s_2+\cdot}$. Moreover, since ζ is white in time, $\mathbf{P}_N \Psi_{s,s+\cdot}$ is independent of $\mathcal{F}_s$ and so are its renormalized powers as those are polynomial functions of $\mathbf{P}_N \Psi_{s,s+\cdot}$, the convergence in the truncation parameter N then gives the result. ▷

Expanding the powers via the binomial formula, we can also get the following expansion in a similar manner as in [TW18]

Lemma IV.2 – *For any* $t, h > 0$*, the following holds true almost surely*

$$\overset{k}{\vee}_{0,t+h} = \sum_{j=0}^{k} \binom{k}{j} \big(e^{-h\mathcal{H}}\Psi_{0,t}\big)^j \overset{k-j}{\vee}_{t,t+h}.$$

Getting our focus back to equation (IV.1), we deal with initial conditions in the support of $\mu^{\mathcal{H}}$, this means $u_0 \in \mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$, as we expect measure invariance properties. Following the usual Da Prato-Debussche argument, we search for solutions under the form $u = \Psi_0 + v$ where v solves the non-linear dynamic

$$\partial_t v + \mathcal{H}v + \sum_{j=0}^{3} \binom{3}{j} v^j \overset{3-j}{\bullet\!\!\vee\!\!\bullet}_{0,\cdot} = 0$$

starting from u_0, where $\overset{k}{\bullet\!\!\vee\!\!\bullet}$ is defined as before. In contrast with [MW17b] and [EMR24] and similar to [TW18], v starts with the same initial initial as u, this somehow makes the analysis easier as far as global well-posedness and Markovian properties are concerned. Grouping the noise terms together as a vector we will refer to as *the noise data* $\mathbf{z} = (\mathbf{z}^{(0)}, \cdots, \mathbf{z}^{(3)})$ where

$$\mathbf{z}^{(j)} := \binom{3}{j} \overset{j}{\bullet\!\!\vee\!\!\bullet}_{0,\cdot}, \tag{IV.4}$$

we may also need a shifted version of $\mathbf{z}$, not starting from 0

$$\mathbf{z}^{(j)}_{t,t+\cdot} := \binom{3}{j} \overset{j}{\bullet\!\!\vee\!\!\bullet}_{t,t+\cdot}. \tag{IV.5}$$

The equation then writes

$$\begin{cases} \partial_t v + \mathcal{H}v + \sum_{j=0}^{3} v^j \mathbf{z}^{(3-j)} = 0, \\ v_{t=0} = u_0. \end{cases} \tag{IV.6}$$

In view of the Schauder estimates Proposition I.6 for $\mathcal{H}$ and given some $\varepsilon, \sigma, T > 0$ we expect v to belong to the space

$$X_T^{-\varepsilon,\sigma} = C\Big([0,T], \mathcal{C}^{-\varepsilon}(\mathbb{T}^2)\Big) \cap C\Big((0,T], \mathcal{C}^{\sigma}(\mathbb{T}^2)\Big)$$

equiped with the norm

$$\|v\|_{X_T^{-\varepsilon,\sigma}} := \|v\|_{L^\infty_T \mathcal{C}^{-\varepsilon}} + \sup_{0<t\leq T} t^{\frac{\sigma+\varepsilon}{2}} \|v_t\|_{\mathcal{C}^\sigma}$$

that takes the blow-up at $t = 0$ into account. The precise condition on ε, σ will be described below. We emphasize that we will end up with the restriction $\sigma < 1$ instead of $\sigma < 2$ as is the case for the Laplace operator instead of $\mathcal{H}$, this is due to the singular nature of $\mathcal{H}$, that is, $\mathcal{H}$-based Sobolev spaces do not provide better regularity than $\mathcal{C}^{1^-}$ (see Chapter I for the full story).

IV.1.1 About global well-posedness

In the next section, we will mostly be concerned with wether the local solutions to (IV.1) exist globally or not. In Chapter III we proved almost-sure global existence for initial data given by the invariant measure. We enhance this result into deterministic global existence for any initial data in $\mathcal{C}^{-\varepsilon}$. We begin by a naive but instructive energy bound for solutions to (IV.6).

Lemma IV.3 – *Let $u_0 \in \mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$ and v be the solution to (IV.6) starting from u_0 up to time $T > 0$, and fix $0 < t_0 < t_1 < T$. Then the energy*

$$\psi(t) := \frac{1}{2}\|v(t)\|_{L^2}^2 + \int_0^t \|v(s)\|_{\mathcal{D}^1}^2 \, ds$$

is well-defined for $t \in (t_0, t_1)$ and there exists a constant $C > 0$ that depends only on the noise data $\mathbf{z}$ and t_0 such that

$$\sup_{t_0 < t < t_1} \psi(t) \leq C.$$

Proof – This follows from the very same computations as in the proof of Proposition III.14, without the regularizing operator $\mathbf{P}_N$ whose only purpose was to obtain a bound in $\mathcal{C}^{-\varepsilon}$ from a bound in L^2, which we cannot achieve here. ▷

This is however insufficient to ensure global existence for the solution to (IV.6) as we only get an L^2 bound, which does not guarantee a $\mathcal{C}^{-\varepsilon}$ bound in the absence of a regularizing operator $\mathbf{P}_N$. However from Besov embedding and compararison with Sobolev spaces in Lemma A.3 it would be enough to obtain a bound in some L^p for p large enough.

IV.1.2 Raw idea to obtain global solutions : L^{3p-2} bound

Following the mild formulation of the equation, at each positive time v is almost a $\mathcal{D}^2$ function. This in particular ensures that v has a paracontrolled structure

$$v = v \otimes X + w$$

for some remainder w whose regularity can be evaluated in usual spaces. To simplify the analysis of the cubic terms, we further introduce a truncation parameter n and rather consider the ansatz

$$v = v \otimes X_{>n} + w^{(n)}$$

where $X_{>n} = \Delta_{>n} X$. Following the construction of the operator as per usual (see for instance Chapter I, or [Mou22] and [BDM23] for the full story), we are equiped with the one-to-one mapping $\Gamma_n : w^{(n)} \mapsto v$, defined as the inverse mapping of $\Gamma_n^{-1} : v \mapsto v - v \otimes X_{>n}$. It is worth noting that $w^{(n)}$ inherits the same regularity properties as w, uniformly in n. Also, the construction of $\mathcal{H}$ relies heavily on the fact that one can tune the paraproduct $\otimes$ so that v is controlled by w in appropriate norms (the converse being always true), see Lemma I.6. We claim that it is still the case for $w^{(n)}$ and that the bound holds uniformly in n.

Lemma IV.4 – *For any $n \geq 1$ and $p > 16$, there is a constant $C > 0$ that does not depend on n such that*

$$C^{-1}\|v\|_{L^p} \leq \|w_n\|_{L^p} \leq C\|v\|_{L^p}.$$

Proof – This is a direct consequence of Lemma I.6 and the fact that $\|X_{>n}\|_{B^{1-\kappa}_{q,\infty}}$ is bounded by $\|X\|_{B^{1-\kappa}_{q,\infty}}$ uniformly in n, for any $\kappa > 0$ and $q \geq 1$. ▷

The idea behind introducing this new truncation is that we will be able to make cubic terms involving $X_{>n}$ small when n goes to $+\infty$. We then have a correction operator $R_n : B^{1+\kappa}_{p,\infty} \to B^{-\kappa'}_{p,\infty}$ such that

$$\Gamma_n^{-1}\mathcal{H}\Gamma_n w^{(n)} = -\Delta w^{(n)} + \xi \circledcirc w^{(n)} + R_n(w^{(n)}).$$

The dependency of R_n on n will be harmless as R_n is a linear term that will be made small using Young's estimate, no matter the value of n. Equation (IV.6) then becomes (drop the subscript n in $w^{(n)}$)

$$\partial_t w - \Delta w + \xi \circledcirc w + w^3 + Q(v, w, \mathbf{z}) + R_n(w) = 0 \tag{IV.7}$$

where

$$\begin{aligned}
&Q(v, \mathbf{z}) = Q_1(v, w) + Q_2(v, \mathbf{z}) + Q_3(v, \mathbf{z}),\\
&Q_1(v, w) = -v^3 \circledcirc X_{>n} + (v \circledcirc X_{>n})^3 + 3(v \circledcirc X_{>n})w^2 + 3(v \circledcirc X_{>n})^2 w,\\
&Q_2(v, \mathbf{z}) = v^2\mathbf{z}^{(1)} + v\mathbf{z}^{(2)} + \mathbf{z}^{(3)},\\
&Q_3(v, \mathbf{z}) = -(v^2\mathbf{z}^{(1)} + v\mathbf{z}^{(2)} + \mathbf{z}^{(3)}) \circledcirc X_{>n}.
\end{aligned}$$

We will keep v in our notations to make things easier to read, but one should really keep in mind that Γ_n is a one-to-one mapping between w and v so that we could drop the explicit dependency on v and replace it by $\Gamma_n w$. It is crucial noting that the leading power w^3 comes with a positive sign as we intend to test the equation against w^{3p-3}. Testing (IV.7) against w^{3p-3} and integrating in time then yields (assume for the sake of the exposition that $w_0 \in L^{3p-2}$)

$$\begin{aligned}
\frac{1}{3p-2}\Big(\|w_t\|^{3p-2}_{L^{3p-2}} - \|w_0\|^{3p-2}_{L^{3p-2}}\Big) &+ (3p-3)\int_0^t \||\nabla w_s|^2 w_s^{3p-4}\|_{L^1}\, ds + \int_0^t \|w_s\|^{3p}_{L^{3p}}\, ds\\
&= -\int_0^t \langle \xi \circledcirc w_s, w_s^{3p-3}\rangle\, ds - \int_0^t \langle Q(v_s, w_s, \mathbf{z}_s), w_s^{3p-3}\rangle\, ds\\
&\quad - \int_0^t \langle R_n(w_s), w_s^{3p-3}\rangle\, ds.
\end{aligned}$$

Do note that w being the remainder in the paracontrolled expansion of v, w has enough regularity for ∇w to make sense so that $\||\nabla w_s|^2 w_s^{3p-4}\|_{L^1}$ is a *good* (non-negative) term since p is an even integer. Note that Q gathers non-linear contributions of v, Q_1 has a cubic dependency on v, but only through the regularizing operator $\cdot \circledcirc X_{>n}$. As for Q_2 and Q_3, they only involve lower order powers of v, while R on the other hand only depends linearly on v.

The noise ξ being of Hölder regularity $-1-\kappa$ for any $\kappa > 0$, estimating the first integral requires (for instance) $1+2\kappa$ spatial regularity on w :

$$\left|\int_0^t \langle \xi \circledcirc w_s, w_s^{3p-3}\rangle\, ds\right| \lesssim \int_0^t \|w_s\|^{3p-3}_{L^{3p}} \|w_s\|_{B^{1+2\kappa}_{p,\infty}}\, ds$$

$$\lesssim \left(\int_0^t \|w_s\|_{L^{3p}}^{3p}\, ds\right)^{\frac{p-1}{p}} \left(\int_0^t \|w_s\|_{B^{1+2\kappa}_{p,\infty}}^{p}\, ds\right)^{\frac{1}{p}}.$$

As for the contribution from Q_1, we use yet again Hölder inequality as follows :

$$\left|\int_0^t \langle Q_1(v_s,w_s), w_s^{3p-3}\rangle\, ds\right| \lesssim \|\cdot \varolessthan X_{>n}\|_{L^p\to L^p}\int_0^t \|v_s\|_{L^{3p}}^3 \|w_s\|_{L^{3p}}^{3p-3}\, ds$$
$$+\sum_{k=1}^{3}\|\cdot \varolessthan X_{>n}\|_{L^{3p}\to L^{3p}}^k \int_0^t \|v_s\|_{L^{3p}}^k \|w_s\|_{L^{3p}}^{3p-k}\, ds.$$

The key idea is that the higher regularity norms of w only appear integrated in time, we can then leverage the issue of estimating these terms by using the mild formulation of the equation

$$w_t = e^{t\Delta}w_0 - \int_0^t e^{(t-s)\Delta}\Big(\xi \,\circledleq\, w_s + w_s^3 + Q(v_s,w_s,\mathbf{z}_s) + R_n(w_s)\Big)\, ds$$

and the fact that $e^{(t-s)\Delta}$ is a regularizing operator, at the price of a diverging power of $t-s$. From there we will be able to close the estimate and obtain a *coming down from infinity* estimate.

Remark 8 – *This whole Chapter is written in the framework of the* 2*-dimensional torus* $\mathbb{T}^2$ *for the sake of clarity, but the tools we have at hand work exactly the same in the setting of a closed Riemannian surface.*

IV.2 – Solution theory

Local well-posedness theory for equation (IV.6) falls under the scope of the previous chapter, from which we recall the result.

Proposition IV.5 – *Let* $0<\varepsilon<\sigma<1$ *and* $1\leq q<\infty$ *be such that* $\frac{\sigma+\varepsilon}{2}\frac{q}{q-1}<\frac{1}{4}$. *Then there exists* $\theta, C>0$ *such that for any* $T_0, r, R>0$, *there exists*

$$T \propto \min\Big(T_0; \big(\frac{r}{(1+r+R)^3}\big)^{\theta}\Big) \in (0; T_0\wedge 1]$$

such that for any $u_0\in\mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$ *and* $\mathbf{z}\in L^q([0;T_0],\mathcal{C}^{-\varepsilon}(\mathbb{T}^2))^4$ *satisfying* $\|u_0\|_{\mathcal{C}^{-\varepsilon}}\leq r$ *and* $\|\mathbf{z}\|_{L^q_{T_0}\mathcal{C}^{-\varepsilon}}\leq R$, *there exists a unique solution* v *in* $X_T^{-\varepsilon,\sigma}$ *to* (IV.6). *Moreover* v *depends Lipschitz continuously on both* v_0 *and* $\mathbf{z}$, *and satisfies*

$$\|v\|_{X_T^{-\varepsilon,\sigma}} \leq Cr. \tag{IV.8}$$

Setting $u = \Psi_{0,\cdot} + v$, we then obtain local well-posedness of equation (IV.1).

Theorem IV.6 – *Let* $0<\varepsilon<\frac{1}{3}$, $\varepsilon<\sigma<1$, *and* $q>1$ *be such that* $\frac{\sigma+\varepsilon}{2}\frac{q}{q-1}<\frac{1}{3}$. *Then*

$\mathbb{P}$*-almost surely, for any* $u_0 \in \mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$ *and* $T_0 > 0$*, there exists* $T \in (0; T_0 \wedge 1]$ *such that* (IV.1) *admits a solution* $u \in C([0;T]; \mathcal{C}^{-\varepsilon}(\mathbb{T}^2))$*, unique in the affine space* $\mathord{\rotatebox[origin=c]{180}{!}}_{0,\cdot} + X_T^{-\varepsilon,\sigma}$.

IV.2.1 A priori estimate

We now turn to the globalization problem for (IV.1), which amounts to prove global existence for the solution v to (IV.6). In view of the blow-up criterion provided by Proposition IV.5, we have to prove that the $\mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$ norm of v stays bounded uniformly in time, in view of the Besov embedding $L^{3p-2} \hookrightarrow \mathcal{C}^{-\varepsilon}$ it thus suffices to bound the L^{3p-2} norm of v for some even integer p large enough ($p > \frac{4}{3\varepsilon}$ is enough). We give rigorous estimates on the integrand described before to make sense of the previous discussion and obtain an a priori estimate on w. Remember that p is an even integer supposed to be large enough, and T is the maximal time of existence provided by the local existence Theorem IV.5.

Lemma IV.7 – *For any* $\kappa > 0$ *and* $0 < s < t < T$

$$\left| \int_s^t \langle \xi \circledcirc w_r, w_r^{3p-3} \rangle \, dr \right| \lesssim \left(\int_s^t \|w_r\|_{L^{3p}}^{3p} \, dr \right)^{\frac{p-1}{p}} \left(\int_s^t \|w_r\|_{B^{1+2\kappa}_{p,\infty}}^{p} \, dr \right)^{\frac{1}{p}}$$

where the implicit constant depends only on p *and* κ.

Proof – Since ξ has regularity $\mathcal{C}^{-1-\kappa}$ for any $\kappa > 0$, this is a direct consequence of Hölder inequality and the estimate on paraproducts

$$\|\xi \circledcirc w\|_{L^p} \lesssim \|w\|_{B^{1+2\kappa}_{p,\infty}}.$$

$\triangleright$

Lemma IV.8 – *For any* $0 < s < t < T$,

$$\left| \int_s^t \langle R_n(w_r), w_r^{3p-3} \rangle \, dr \right| \lesssim_n \left(\int_s^t \|w_r\|_{L^{3p}}^{3p} \, dr \right)^{\frac{p-1}{p}} \left(\int_s^t \|w_r\|_{B^{1+\varepsilon}_{p,\infty}}^{p} \, dr \right)^{\frac{1}{p}}.$$

Proof – Recall from [BDM23, Lemma 20] that the corrector term is a bounded operator

$$R_n : B^{1+\kappa}_{p,\infty} \to B^{-\kappa'}_{p,\infty}$$

for arbitrary $\kappa, \kappa' > 0$. This yields

$$\begin{aligned}
\left| \langle R_n(w), w^{3p-3} \rangle \right| &\lesssim \|R_n(w)\|_{B^{-\kappa}_{\frac{3p}{2},1}} \|w^{3p-3}\|_{B^{\kappa}_{\frac{3p}{3p-2},\infty}} \\
&\lesssim \|R_n(w)\|_{B^{-\kappa}_{\frac{3p}{2},\infty}} \|w^{3p-3}\|_{B^{\kappa}_{\frac{3p}{3p-2},\infty}} \\
&\lesssim_n \|w\|_{B^{1+\kappa}_{\frac{3p}{2},\infty}} \|w^{3p-3}\|_{B^{\kappa}_{\frac{3p}{3p-2},\infty}}.
\end{aligned}$$

Non-linear estimate from Lemma A.5 allows to handle w^{3p-3}

$$\begin{aligned}\|w^{3p-3}\|_{B^{\kappa}_{\frac{3p}{3p-2},\infty}} &\lesssim \|w^{3p-4}\|_{L^{\frac{3p}{3p-4}}}\|w\|_{B^{\kappa}_{\frac{3p}{2},\infty}}\\ &\lesssim \|w\|^{3p-4}_{L^{3p}}\|w\|_{B^{\kappa}_{\frac{3p}{2},\infty}}.\end{aligned}$$

The goal is now to obtain a similar estimate as above. Having noted that

$$1 = \frac{p-1}{p} + \frac{1}{p} = \frac{3p-3}{3p} + \frac{1}{p}$$

and that for any regularity index $\alpha \in \mathbb{R}$ and $p > \frac{4}{3\varepsilon}$ Lemma A.3 yields

$$B^{\alpha+\frac{\varepsilon}{2}}_{p,\infty} \hookrightarrow B^{\alpha+\frac{\varepsilon}{2}-\frac{2}{3p}}_{\frac{3p}{2},\infty} \hookrightarrow B^{\alpha}_{\frac{3p}{2},\infty}$$

we interpolate $B^{\kappa}_{\frac{3p}{2},\infty}$ between $L^{\frac{3p}{2}}$ and $B^{\sqrt{\kappa}}_{\frac{3p}{2},\infty}$ using Lemma A.4 (remember that $\kappa > 0$ is arbitrarily small, in particular $\kappa < 1$).

$$\|w\|_{B^{\kappa}_{\frac{3p}{2},\infty}} \lesssim \|w\|^{\sqrt{\kappa}}_{B^{\sqrt{\kappa}}_{\frac{3p}{2},\infty}}\|w\|^{1-\sqrt{\kappa}}_{L^{\frac{3p}{2}}}.$$

We do the same for the term with 1^+ regularity

$$\|w\|_{B^{1+\kappa}_{\frac{3p}{2},\infty}} \lesssim \|w\|^{1-\sqrt{\kappa}}_{B^{1+\tilde{\kappa}}_{\frac{3p}{2},\infty}}\|w\|^{\sqrt{\kappa}}_{L^{\frac{3p}{2}}}$$

where $1+\tilde{\kappa} = \frac{1+\kappa}{1-\sqrt{\kappa}} > 1+\kappa$. Taking κ small enough so that $1+\tilde{\kappa} = 1+\frac{\varepsilon}{2}$, the estimate follows by Hölder inequality

$$\begin{aligned}\Big|\int_s^t \langle R_n(w_r), w_r^{3p-3}\rangle\, dr\Big| &\lesssim_n \int_s^t \|w_r\|^{3p-3}_{L^{3p}}\|w_r\|_{B^{1+\varepsilon}_{p,\infty}}\, dr\\ &\lesssim_n \left(\int_s^t \|w_r\|^{3p}_{L^{3p}}\, dr\right)^{\frac{p-1}{p}}\left(\int_s^t \|w_r\|^{p}_{B^{1+\varepsilon}_{p,\infty}}\, dr\right)^{\frac{1}{p}}.\end{aligned}$$

▷

Lemma IV.9 – *For any $\delta > 0$, there exists $n_0(\delta, p)$ such that for any $n \geq n_0$ and* $0 < s < t < T$

$$\left|\int_s^t \langle Q_1(v_r, w_r), w_r^{3p-3}\rangle\, dr\right| \leq \delta \int_s^t \|w_r\|^{3p}_{L^{3p}}\, dr.$$

Proof – Fix $\delta > 0$, as Q_1 gathers cubic contributions that involve a $X_{>n}$ term, multiple uses of Hölder inequality yield

$$\left|\int_s^t \langle Q_1(v_r, w_r), w_r^{3p-3}\rangle \, dr\right| \lesssim \|\cdot \circledcirc X_{>n}\|_{L^p\to L^p} \int_s^t \|v_r\|_{L^{3p}}^3 \|w_r\|_{L^{3p}}^{3p-3} \, dr + \sum_{k=1}^{3} \|\cdot \circledcirc X_{>n}\|_{L^{3p}\to L^{3p}}^k \int_s^t \|v_r\|_{L^{3p}}^k \|w_r\|_{L^{3p}}^{3p-k} \, dr.$$

The above implicit constant does not depend on n. Recall from Lemma IV.4 that v is controlled by w, uniformly in n. Thus, choosing n large enough (depending on δ and p),

$$\left|\int_s^t \langle Q_1(v_r, w_r), w_r^{3p-3}\rangle \, dr\right| \leq \delta \int_s^t \|w_r\|_{L^{3p}}^{3p} \, dr.$$

$\triangleright$

For the other non-linear terms, note that only the regularizing properties of $\cdot \circledcirc X_{>n}$ are used. As such, the estimates all end up being uniform in the truncation parameter n.

Lemma IV.10 – *For any $\delta > 0$, there exists a constant $c > 0$ that depends only on δ, p, ε such that for any $0 < s < t < T$*

$$\left|\int_s^t \langle Q_2(v_r, \mathbf{z}_r), w_r^{3p-3}\rangle \, dr\right| \leq \delta \int_s^t \|w_r\|_{L^{3p}}^{3p} \, dr + cK_t\Big(1 + \int_s^t \|w_r\|_{B^{2\varepsilon}_{p,\infty}}^p \, dr\Big)$$

where the constant K_t is defined by

$$K_t = \sup_{0\leq r\leq t} \|\mathbf{z}_r^{(1)}\|_{\mathcal{C}^{-\varepsilon}}^{2p} + \int_0^t \|\mathbf{z}_r^{(2)}\|_{\mathcal{C}^{-\varepsilon}}^{3p} \, dr + \int_0^t \|\mathbf{z}_r^{(3)}\|_{\mathcal{C}^{-\varepsilon}}^{3p} \, dr. \tag{IV.9}$$

Proof – The major issue with Q_2 is handling the quadratic part $v^2\mathbf{z}^{(1)}$, to that end we further expand Q_2 as

$$Q_2 = (v \circledcirc X_{>n})^2\mathbf{z}^{(1)} + w^2\mathbf{z}^{(1)} + 2w(v \circledcirc X_{>n})\mathbf{z}^{(1)} + (v \circledcirc X_{>n})\mathbf{z}^{(2)} + w\mathbf{z}^2 + \mathbf{z}^3$$

to take advantage of the regularizing effect of $\cdot \circledcirc X_{>n}$.
Starting with the w^2 term, as $\mathbf{z}^{(1)}$ has spatial regularity $-\varepsilon$ we get

$$\left|\langle w^2\mathbf{z}^{(1)}, w^{3p-3}\rangle\right| \lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|w^{3p-1}\|_{B^{\varepsilon}_{1,\infty}}.$$

Using the power product rule Lemma A.5, interpolation Lemma A.4 in Besov spaces we bound w^{3p-1} as

$$\begin{aligned}\|w^{3p-1}\|_{B^{\varepsilon}_{1,\infty}} &\lesssim \|w^{3p-2}\|_{L^{\frac{3p}{3p-2}}} \|w\|_{B^{\varepsilon}_{\frac{3p}{2},\infty}} \\ &\lesssim \|w\|_{L^{3p}}^{3p-\frac{3}{2}} \|w\|_{B^{2\varepsilon}_{p,\infty}}^{\frac{1}{2}}.\end{aligned}$$

We then bound $\left|\int_s^t \langle w_r^2\mathbf{z}_r^{(1)}, w_r^{3p-3}\rangle \, dr\right|$ by integrating over r knowing $\mathbf{z}^{(1)}$ is in $C_t\mathcal{C}^{-\varepsilon}$

and using Young's estimate

$$\left|\int_s^t \langle w_r^2 \mathbf{z}_r^{(1)}, w_r^{3p-3}\rangle\, dr\right| \le \int_s^t \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^{3p-\frac{3}{2}} \|w\|_{B^{2\varepsilon}_{p,\infty}}^{\frac{1}{2}}\, dr$$
$$\le \delta \int_s^t \|w_r\|_{L^{3p}}^{3p}\, dr + \Big(\sup_{0\le r\le t} \|\mathbf{z}_r^{(1)}\|_{\mathcal{C}^{-\varepsilon}}\Big)^p c(\delta) \int_s^t \|w_r\|_{B^{2\varepsilon}_{p,\infty}}^p\, dr.$$

Turning to the $(v \circledcirc X_{>n})^2$ term, we make use of the regularizing effects of the operator $\cdot \circledcirc X_{>n}$

$$\begin{aligned}\left|\langle (v\circledcirc X_{>n})^2 \mathbf{z}^{(1)}, w^{3p-3}\rangle\right| &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|(v\circledcirc X_{>n})^2\|_{\mathcal{C}^{\varepsilon}} \|w^{3p-3}\|_{B^{\varepsilon}_{1,\infty}}\\ &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|v\|_{L^{3p}}^2 \|w^{3p-3}\|_{B^{\varepsilon}_{1,\infty}}\\ &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^2 \|w^{3p-3}\|_{B^{\varepsilon}_{1,\infty}}.\end{aligned}$$

As before, the power product and interpolation rules yield

$$\begin{aligned}\|w^{3p-3}\|_{B^{\varepsilon}_{1,\infty}} &\lesssim \|w^{3p-4}\|_{L^{\frac{3p}{3p-4}}} \|w\|_{B^{\varepsilon}_{\frac{3p}{4},\infty}}\\ &\lesssim \|w\|_{L^{3p}}^{3p-4+\frac{1}{2}} \|w\|_{B^{2\varepsilon}_{\frac{3p}{7},\infty}}^{\frac{1}{2}}\end{aligned}$$

and we conclude using once again Young estimate. Same goes for the mixed term $w(v\circledcirc X_{>n})$

$$\begin{aligned}\left|\langle w(v\circledcirc X_{>n}) \mathbf{z}^{(1)}, w^{3p-3}\rangle\right| &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|v\circledcirc X_{>n}\|_{\mathcal{C}^{\varepsilon}} \|w^{3p-2}\|_{B^{\varepsilon}_{1,\infty}}\\ &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|v\|_{L^{3p}} \|w^{3p-2}\|_{B^{\varepsilon}_{1,\infty}}\\ &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}} \|w^{3p-2}\|_{B^{\varepsilon}_{1,\infty}}\\ &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^{3p-2} \|w\|_{B^{\varepsilon}_{p,\infty}}\\ &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^{3p-\frac{3}{2}} \|w\|_{B^{2\varepsilon}_{\frac{3p}{5},\infty}}^{\frac{1}{2}}\end{aligned}$$

and we finally obtain that

$$\left|\int_s^t \left\langle v_r^2 \mathbf{z}^{(1)}, w_r^{3p-3}\right\rangle\, dr\right| \le \delta \int_s^t \|w_r\|_{L^{3p}}^{3p}\, dr + \Big(\sup_{0\le r\le t} \|\mathbf{z}_r^{(1)}\|_{\mathcal{C}^{-\varepsilon}}\Big)^{2p} c(\delta) \int_s^t \|w_r\|_{B^{2\varepsilon}_{p,\infty}}^p\, dr.$$

The bound with the linear part or the constant part all follow from similar estimates, the only difference is that $\mathbf{z}^{(2)}$ and $\mathbf{z}^{(3)}$ are not continuous in time but rather L^q for any $q < +\infty$. This is however not an issue as the homogeneity in w in these terms is at most 1. For the linear terms, we get

$$\begin{aligned}\left|\langle (v\circledcirc X_{>n}) \mathbf{z}^{(2)}, w^{3p-3}\rangle\right| &\lesssim \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}} \|v\circledcirc X_{>n}\|_{\mathcal{C}^{\varepsilon}} \|w^{3p-3}\|_{B^{\varepsilon}_{1,\infty}}\\ &\lesssim \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}} \|v\|_{L^{3p}} \|w^{3p-3}\|_{B^{\varepsilon}_{1,\infty}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}} \|w^{3p-3}\|_{B^{\varepsilon}_{1,\infty}} \\
&\lesssim \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^{3p-3} \|w\|_{B^{\varepsilon}_{\frac{3p}{4},\infty}} \\
&\lesssim \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^{3p-\frac{5}{2}} \|w\|_{B^{2\varepsilon}_{\frac{3p}{5},\infty}}^{\frac{1}{2}} \\
&\leq \delta \|w\|_{L^{3p}}^{3p} + c(\delta) \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}}^{\frac{6p}{5}} \|w\|_{B^{2\varepsilon}_{p,\infty}}^{\frac{3p}{5}}
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\langle w\mathbf{z}^{(2)}, w^{3p-3}\rangle\right| &\lesssim \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}} \|w^{3p-2}\|_{B^{\varepsilon}_{1,\infty}} \\
&\lesssim \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^{3p-3} \|w\|_{B^{\varepsilon}_{p,\infty}} \\
&\lesssim \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^{3p-\frac{5}{2}} \|w\|_{B^{2\varepsilon}_{p,\infty}}^{\frac{1}{2}} \\
&\leq \delta \|w\|_{L^{3p}}^{3p} + c(\delta) \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}}^{\frac{6p}{5}} \|w\|_{B^{2\varepsilon}_{p,\infty}}^{\frac{3p}{5}}.
\end{aligned}
$$

Integrating over time and applying Young's inequality, we obtain

$$
\left|\int_s^t \left\langle v_r \mathbf{z}_r^{(2)}, w_r^{3p-3}\right\rangle dr\right| \leq \delta \int_s^t \|w_r\|_{3p}^{3p} dr + c(\delta) \int_0^t \|\mathbf{z}_r^{(2)}\|_{\mathcal{C}^{-\varepsilon}}^{3p} dr \Big(1 + \int_s^t \|w_r\|_{B^{2\varepsilon}_{p,\infty}} dr\Big).
$$

As for the linear term, this is basically the same idea, while being slightly easier as we do not need to use an interpolation inequality

$$
\begin{aligned}
\left|\langle \mathbf{z}^{(3)}, w^{3p-3}\rangle\right| &\lesssim \|\mathbf{z}^{(3)}\|_{\mathcal{C}^{-\varepsilon}} \|w^{3p-3}\|_{B^{\varepsilon}_{1,\infty}} \\
&\lesssim \|\mathbf{z}^{(3)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^{3p-4} \|w\|_{B^{\varepsilon}_{p,\infty}} \\
&\lesssim \delta \|w\|_{L^{3p}}^{3p} + c(\delta) \|\mathbf{z}^{(3)}\|_{\mathcal{C}^{-\varepsilon}}^{\frac{3p}{4}} \|w\|_{B^{\varepsilon}_{p,\infty}}^{\frac{3p}{4}}.
\end{aligned}
$$

Altogether, we obtain

$$
\left|\int_s^t \left\langle Q_2(v_r, \mathbf{z}_r), w_r^{3p-3}\right\rangle dr\right| \leq \delta \int_s^t \|w_r\|_{L^{3p}}^{3p} dr + c(\delta) K_t \Big(1 + \int_s^t \|w_r\|_{B^{2\varepsilon}_{p,\infty}}^{p} dr).
$$

▷

Lemma IV.11 – *For any $\delta > 0$, there exists a constant $c > 0$ that depends only on δ, p, ε such that for any $0 < s < t < T$*

$$
\left|\int_s^t \langle Q_2(v_r, \mathbf{z}_r), w_r^{3p-3}\rangle\, dr\right| \leq \delta \int_s^t \|w_r\|_{L^{3p}}^{3p} dr + cK_t \Big(1 + \int_s^t \|w_r\|_{B^{2\varepsilon}_{p,\infty}}^{p} dr\Big)
$$

where K_t is defined in (IV.9).

Proof – This is an easier counterpart of the previous proof as the regularizing operator $\cdot \circledcirc X_{>n}$ allows to estimate w^{3p-3} in $L^{\frac{3p}{3p-3}}$ directly

$$\langle Q_3(v,\mathbf{z}), w^{3p-3}\rangle \lesssim \|v^2\mathbf{z}^{(1)} + v\mathbf{z}^{(2)} + \mathbf{z}^{(3)}\|_{\mathcal{C}^{-\varepsilon}} \|w\|_{L^{3p}}^{3p-3}$$

and the first term on the right-hand side is dealt with as before. ▷

All in all, we obtain a bound for w as follows.

Proposition IV.12 – *For any even integer $p > \frac{4}{3\varepsilon}$ and n large enough depending on p, $0 < s < t < T$ we have*

$$\|w_t\|_{L^{3p-2}}^{3p-2} + \int_s^t \|w_r\|_{L^{3p}}^{3p}\, dr \lesssim \|w_s\|_{L^{3p-2}}^{3p-2} + K_t\Big(1 + \int_s^t \|w_r\|_{B^{1+\varepsilon}_{p,\infty}}^p\, dr\Big)$$

where the implicit constant depends only on p and ε.

IV.2.2 Higher regularity estimates on w

As we intend to use a *coming down from infinity* argument, it remains to bound the rightmost term in order to close the estimate. This follows from the mild formulation of the equation.

Lemma IV.13 – *For any $0 < s < t < T$*

$$\int_s^t \|w_r\|_{B^{1+4\varepsilon}_{p,\infty}}^p\, dr \lesssim 1 + \int_s^t \|e^{(r-s)\Delta} w_s\|_{B^{1+4\varepsilon}_{p,\infty}}^p\, dr + \tilde{K}_t \int_s^t \|w_r\|_{L^{3p}}^{3p}\, dr.$$

The constant $\tilde{K}_t$ has the following form

$$\tilde{K}_t = c\Big(\sup_{0\le r\le t} \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} + t^{\mathfrak{a}}\|\mathbf{z}^{(2)}\|_{L^{\mathfrak{q}}_t\mathcal{C}^{-\varepsilon}} + t^{\mathfrak{a}}\|\mathbf{z}^{(3)}\|_{L^{\mathfrak{q}}_t\mathcal{C}^{-\varepsilon}}\Big)^{\mathfrak{b}} \tag{IV.10}$$

for some constant $c > 0$ and exponents $\mathfrak{a}, \mathfrak{b} > 0$ and $\mathfrak{q} > 1$ all depending only on p and ε.

Remark 9 – *In the following, the exact definition of $\tilde{K}$ may differ from one line to the other, but we still write $\tilde{K}$ for a constant having the same form as (IV.10). Note that $\tilde{K}_t$ depends on the randomness of $\mathbf{z}$ only on the interval $[0,t]$, this will come in handy later on.*

Proof – Set $\gamma = 1 + 4\varepsilon$, writing the mild formulation

$$w_t = e^{(t-s)\Delta} w_s - \int_s^t e^{(t-r)\Delta}\left(\xi \circledcirc w_r + R_n(w_r) + w_r^3 + Q(v_r, w_r, \mathbf{z}_r)\right) dr$$

we take advantage of the regularizing effects of the heat operator $e^{(t-r)\Delta}$. Provided p is large enough, and for any $\kappa < \frac{\gamma}{2}$

$$\int_s^t \|e^{(t-r)\Delta}\big(\xi \circledcirc w_r\big)\|_{B^{\gamma}_{p,\infty}}\, dr \lesssim \left(\int_s^t (t-r)^{-\frac{\gamma p}{2(p-1)}}\, dr\right)^{\frac{p-1}{p}} \left(\int_s^t \|w_r\|_{B^{1+\kappa}_{p,\infty}}^p\, dr\right)^{\frac{1}{p}}$$

and

$$\int_s^t \|e^{(t-r)\Delta} R_n(w_r)\|_{B^\gamma_{p,\infty}}\, dr \lesssim \left(\int_s^t (t-r)^{-\frac{(\gamma+\kappa)p}{2(p-1)}}\, dr\right)^{\frac{p-1}{p}} \left(\int_s^t \|w_r\|^p_{B^{1+\kappa}_{p,\infty}}\, dr\right)^{\frac{1}{p}}.$$

For the cubic part we get

$$\int_s^t \|e^{(t-r)\Delta} w_r^3\|_{B^\gamma_{p,\infty}}\, dr \lesssim \left(\int_s^t (t-r)^{-\frac{\gamma p}{2(p-1)}}\, dr\right)^{\frac{p-1}{p}} \left(\int_s^t \|w_r\|^{3p}_{L^{3p}}\, dr\right)^{\frac{1}{p}}.$$

Now for the collection of terms appearing in Q, Q_1 gathers cubic homogeneity terms that we bound using Hölder inequality and Lemma IV.4,

$$\begin{aligned}\|Q_1(w)\|_{L^p} &\lesssim \|v^3\|_{L^p} + \|v\|^3_{L^{3p}} + \|v\|_{L^{3p}}\|w^2\|_{L^{\frac{3p}{2}}} + \|v\|^2_{L^{3p}}\|w\|_{L^{3p}} \\ &\lesssim \|w\|^3_{L^{3p}}.\end{aligned}$$

Integrating against $e^{(t-r)\Delta}$ yields

$$\int_s^t \|e^{(t-r)\Delta} Q_1(v_r,w_r)\|_{B^\gamma_{p,\infty}}\, dr \lesssim \left(\int_s^t (t-r)^{-\frac{\gamma p}{2(p-1)}}\, dr\right)^{\frac{p-1}{p}} \left(\int_s^t \|w_r\|^{3p}_{L^{3p}}\, dr\right)^{\frac{1}{p}}.$$

Making use of the product rule Lemma A.5 to handle the quadratic terms, we obtain for Q_2

$$\begin{aligned}\|Q_2(v,\mathbf{z})\|_{B^{-\varepsilon}_{p,\infty}} &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}}\|v\|_{B^{2\varepsilon}_{\frac{3p}{2},\infty}}\|v\|_{L^{3p}} + \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}}\|v\|_{B^{2\varepsilon}_{p,\infty}} + \|\mathbf{z}^{(3)}\|_{\mathcal{C}^{-\varepsilon}} \\ &\lesssim \|\mathbf{z}^{(1)}\|_{\mathcal{C}^{-\varepsilon}}\|w\|_{B^{2\varepsilon}_{\frac{3p}{2},\infty}}\|w\|_{L^{3p}} + \|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}}\|w\|_{B^{2\varepsilon}_{p,\infty}} + \|\mathbf{z}^{(3)}\|_{\mathcal{C}^{-\varepsilon}}.\end{aligned}$$

The analysis of the two last terms on the right-hand side is straightforward, we investigate the quadratic part. Integrating against $e^{(t-r)\Delta}$, we first use Hölder estimate

$$\begin{aligned}\int_s^t \|e^{(t-r)\Delta}\mathbf{z}^{(1)}_r v_r^2\|_{B^\gamma_{p,\infty}}\, dr &\lesssim \int_s^t (t-r)^{-\frac{\gamma+\varepsilon}{2}}\|\mathbf{z}^{(1)}_r\|_{\mathcal{C}^{-\varepsilon}}\|w_r\|_{B^{2\varepsilon}_{\frac{3p}{2},\infty}}\|w_r\|_{L^{3p}}\, dr \\ &\lesssim \left(\int_s^t (t-r)^{-\frac{3(\gamma+\varepsilon)p}{2(3p-4)}}\|\mathbf{z}^{(1)}_r\|^{\frac{3p}{3p-4}}_{\mathcal{C}^{-\varepsilon}}\, dr\right)^{\frac{3p-4}{3p}} \left(\int_s^t \|w_r\|^p_{B^{2\varepsilon}_{\frac{3p}{2},\infty}}\, dr\right)^{\frac{1}{p}} \left(\int_s^t \|w_r\|^{3p}_{L^{3p}}\, dr\right)^{\frac{1}{3p}}.\end{aligned}$$

We treat the $\mathbf{z}^{(1)}$ integral as follows : note that $\alpha_1 := \frac{3(1+5\varepsilon)p}{2(3p-4)} < 1$ (this is possible provided ε is small enough and p large enough), then

$$\int_s^t (t-r)^{-\frac{3(\gamma+\varepsilon)p}{2(3p-4)}}\|\mathbf{z}^{(1)}_r\|^{\frac{3p}{3p-4}}_{\mathcal{C}^{-\varepsilon}}\, dr \leq \frac{1}{1-\alpha_1}(t-s)^{1-\alpha_1}\sup_{0\leq r\leq t}\|\mathbf{z}^{(1)}_r\|^{\frac{3p}{3p-4}}_{\mathcal{C}^{\varepsilon}}.$$

Note that interpolation Lemma A.4 between L^{3p} and $B^{4\varepsilon}_{p,\infty}$ yields

$$\|w\|_{B^{2\varepsilon}_{\frac{3p}{2},\infty}} \lesssim \|w\|^{\frac{1}{2}}_{L^{3p}} \|w\|^{\frac{1}{2}}_{B^{4\varepsilon}_{p,\infty}}.$$

Thus, finally use Young's estimate to obtain

$$\begin{aligned}\left(\int_s^t \|w_r\|^p_{B^{2\varepsilon}_{\frac{3p}{2},\infty}}\,dr\right)^{\frac{1}{p}} \left(\int_s^t \|w_r\|^{3p}_{L^{3p}}\,dr\right)^{\frac{1}{3p}} &\lesssim \left(\int_s^t \|w_r\|^{\frac{p}{2}}_{B^{4\varepsilon}_{p,\infty}} \|w_r\|^{\frac{p}{2}}_{L^{3p}}\right)^{\frac{1}{p}} \left(\int_s^t \|w_r\|^{3p}_{L^{3p}}\,dr\right)^{\frac{1}{3p}} \\ &\lesssim \left(\int_s^t \|w_r\|^{\frac{3p}{5}}_{B^{4\varepsilon}_{p,\infty}}\,dr\right)^{\frac{5}{6p}} \left(\int_s^t \|w_r\|^{3p}_{L^{3p}}\,dr\right)^{\frac{1}{2p}} \\ &\lesssim \left(\int_s^t \|w_r\|^{p}_{B^{4\varepsilon}_{p,\infty}}\,dr\right)^{\frac{1}{p}} + \left(\int_s^t \|w_r\|^{3p}_{L^{3p}}\,dr\right)^{\frac{1}{p}}.\end{aligned}$$

The very same argument can be used to deal with the lower homogeneity terms of Q_2 and Q_3, although we make use of $\mathbf{z}^{(2)}$ and $\mathbf{z}^{(3)}$ being L^q in time for any $q < +\infty$ to ensure that integrals of the type

$$\int_s^t (t-r)^{-\sigma} \|\mathbf{z}^{(i)}_r\|_{\mathcal{C}^{-\varepsilon}}\,dr$$

for $\sigma \in (0,1)$ and $i \in \{1,2\}$ are finite and uniformly bounded in s by some time integral of $\|\mathbf{z}^{(2)}\|_{\mathcal{C}^{-\varepsilon}}$ and $\|\mathbf{z}^{(3)}\|_{\mathcal{C}^{-\varepsilon}}$. This proves that

$$\|w_t\|_{B^{\gamma}_{p,\infty}} \lesssim 1 + \|e^{(t-s)\Delta} w_s\|_{B^{\gamma}_{p,\infty}} + \tilde{K}_t \left(\left(\int_s^t \|w_r\|^p_{B^{1+\kappa}_{p,\infty}}\,dr\right)^{\frac{1}{p}} + \left(\int_s^t \|w_r\|^{3p}_{L^{3p}}\,dr\right)^{\frac{1}{p}}\right)$$

for any $0 < \gamma < 2$, $\kappa > 0$ $p > 1$ large enough, where the implicit constant depends only on ε, κ, p.
Since $\gamma = 1 + 4\varepsilon$, it follows from interpolation and Young inequalities that

$$\begin{aligned}\|w\|^p_{B^{1+\varepsilon}_{p,\infty}} &\lesssim \|w\|^{\frac{1+\varepsilon}{1+4\varepsilon}p}_{B^{1+4\varepsilon}_{p,\infty}} \|w\|^{\frac{3\varepsilon}{1+4\varepsilon}p}_{L^p} \\ &\lesssim \|w\|^{\frac{1+\varepsilon}{1+3\varepsilon}p}_{B^{1+4\varepsilon}_{p,\infty}} + \|w\|^{3p}_{L^{3p}}.\end{aligned}$$

Setting $\kappa = \varepsilon$ and integrating over time finally yield

$$\int_s^t \|w_r\|^p_{B^{1+4\varepsilon}_{p,\infty}}\,dr \lesssim 1 + \int_s^t \|e^{(r-s)\Delta} w_s\|^p_{B^{1+4\varepsilon}_{p,\infty}}\,dr + \tilde{K}_t \int_s^t \|w_r\|^{3p}_{L^{3p}}\,dr + \int_s^t \|w_r\|^{\frac{1+\varepsilon}{1+3\varepsilon}p}_{B^{1+4\varepsilon}_{p,\infty}}\,dr.$$

Since $\frac{1+\varepsilon}{1+4\varepsilon} < 1$, the last term on the right-hand side can be absorbed by the left-hand side using Young's estimate, which concludes the proof. $\triangleright$

It only remains to turn the previous results into a proper integral inequality, this is the purpose of the following proposition.

Proposition IV.14 – *For any $0 < s < t < T$, $p > \frac{4}{3\varepsilon}$ even integer and n large enough, the following estimate holds*

$$\int_s^t \left(\|w_r\|_{L^{3p-2}}^{3p-2} + \|w_r\|_{B_{p,\infty}^{1+4\varepsilon}}^{\frac{3p-2}{3}} \right)^{\frac{3p}{3p-2}} dr \lesssim 1 + \|w_s\|_{L^{3p-2}}^{3p-2} + \|w_s\|_{B_{p,\infty}^{1+4\varepsilon}}^{\frac{3p-2}{3}}$$

where the implicit constant writes

$$c(p,\varepsilon) + \tilde{K}_t$$

for some $c(p,\varepsilon) > 0$.

Proof – We collect the two main ingredients provided by the previous Lemmas IV.12 and IV.13

$$\|w_t\|_{L^{3p-2}}^{3p-2} + \int_s^t \|w_r\|_{L^{3p}}^{3p}\, dr \lesssim \|w_s\|_{L^{3p-2}}^{3p-2} + \int_s^t \|w_r\|_{B_{p,\infty}^{1+\varepsilon}}^{p}\, dr$$

and

$$\int_s^t \|w_r\|_{B_{p,\infty}^{1+4\varepsilon}}^{p}\, dr \lesssim 1 + \int_s^t \|e^{(r-s)\Delta} w_s\|_{B_{p,\infty}^{1+4\varepsilon}}^{p}\, dr + \int_s^t \|w_r\|_{L^{3p}}^{3p}\, dr.$$

Interpolating $B_{p,\infty}^{1+\varepsilon}$ between L^p and $B_{p,\infty}^{1+4\varepsilon}$ and using Young's estimate, we get that for any $\delta > 0$

$$\|w\|_{B_{p,\infty}^{1+\varepsilon}}^{p} \leq c(\delta) \|w\|_{B_{p,\infty}^{1+4\varepsilon}}^{\frac{1+\varepsilon}{1+4\varepsilon}p} + \delta \|w\|_{L^{3p}}^{3p}$$

and since $\frac{1+\varepsilon}{1+4\varepsilon} < 1$, integrating over time yields

$$\int_s^t \|w_r\|_{B_{p,\infty}^{1+\varepsilon}}^{p}\, dr \leq c + \delta \left(\int_s^t \|w_r\|_{L^{3p}}^{3p} dr + \int_s^t \|w_r\|_{B_{p,\infty}^{1+4\varepsilon}}^{p} dr \right)$$

for some constant $c > 0$ that depends on δ, p and ε. Now set $\gamma := (1+4\varepsilon)\frac{p-1}{p}$ so that

$$\begin{aligned} \|w\|_{B_{p,\infty}^{\gamma}}^{p} &\lesssim \|w\|_{B_{p,\infty}^{1+4\varepsilon}}^{p-1} \|w\|_{L^p} \\ &\lesssim \|w\|_{B_{p,\infty}^{1+4\varepsilon}}^{\frac{3p-2}{3}} + \|w\|_{L^p}^{3p-2}. \end{aligned}$$

From the estimate in Lemma IV.13 we then get

$$\begin{aligned} \int_s^t \|w_r\|_{B_{p,\infty}^{1+\varepsilon}}^{p}\, dr &\leq c + \delta \int_s^t \|w_r\|_{L^{3p}}^{3p} dr + \delta \int_s^t \|e^{(r-s)\Delta} w_s\|_{B_{p,\infty}^{1+4\varepsilon}}^{p}\, dr \\ &\leq c + \delta \int_s^t \|w_r\|_{L^{3p}}^{3p} dr + \delta \|w_s\|_{B_{p,\infty}^{\gamma}}^{p} \\ &\leq c + \delta \int_s^t \|w_r\|_{L^{3p}}^{3p} dr + c(\delta) \Big(\|w_s\|_{B_{p,\infty}^{1+4\varepsilon}}^{\frac{3p-2}{3}} + \|w_s\|_{L^p}^{3p-2} \Big). \end{aligned}$$

Pluging this in the right-hand side of Lemma IV.12, we get

$$\|w_t\|_{L^{3p-2}}^{3p-2} + \int_s^t \|w_r\|_{L^{3p}}^{3p}\,dr + \int_s^t \|w_r\|_{B^{1+4\varepsilon}_{p,\infty}}^{p}\,dr \lesssim 1 + \|w_s\|_{L^{3p-2}}^{3p-2} + \|w_s\|_{B^{1+4\varepsilon}_{p,\infty}}^{\frac{3p-2}{3}}$$

where the implicit constant depends only on ε, p and $\mathbf{z}$. This will be used either as

$$\|w_t\|_{L^{3p-2}}^{3p-2} \lesssim 1 + \|w_s\|_{L^{3p-2}}^{3p-2} + \|w_s\|_{B^{1+4\varepsilon}_{p,\infty}}^{\frac{3p-2}{3}} \tag{IV.11}$$

or

$$\int_s^t \left(\|w_r\|_{L^{3p-2}}^{3p-2} + \|w_r\|_{B^{1+4\varepsilon}_{p,\infty}}^{\frac{3p-2}{3}}\right)^{\frac{3p}{3p-2}} dr \lesssim 1 + \|w_s\|_{L^{3p-2}}^{3p-2} + \|w_s\|_{B^{1+4\varepsilon}_{p,\infty}}^{\frac{3p-2}{3}},$$

this last one being the claimed estimate. ▷

IV.2.3 Global well-posedness and moments estimate

Recall the following comparison lemma from [MW17b]

Lemma IV.15 – *Let $0 < s < \tau$, $c > 0$ and $F : [s, \tau) \to \mathbb{R}_+$ be continuous such that for any $s < s_1 < s_2 < \tau$*

$$\int_{s_1}^{s_2} F^\lambda(r)\,dr \lesssim cF(s_1)$$

for some $\lambda > 1$ and $c < 0$. Then there exists $N \in \mathbb{N}^*$ and $s = t_0 < t_1 < \cdots < t_N = \tau$ such that

$$F(t_n) \lesssim c^{\frac{1}{\lambda-1}}(t_{n+1} - s)^{-\frac{1}{\lambda-1}}$$

where the implicit constant depends only on λ.

This lemma ensures that the behavior of F away from 0 is controlled uniformly in the initial condition, this is the so-called *coming down from infinity.*

Proposition IV.16 – *For any even integer $p > \frac{4}{3\varepsilon}$ and n large enough*

$$\forall 0 < t < T\,,\ (1 \wedge \sqrt{t})\|w_t\|_{L^{3p-2}} \le \tilde{K}_t.$$

As such, the solution to (IV.6) is global in time. Moreover, the the right-hand side does not depend on u_0 and only depends on the randomness of $\mathbf{z}$ on the time interval $[0, t]$.

Proof – Let $0 < s < t < T$ and consider

$$F : r \in [s, T) \mapsto \|w_r\|_{L^{3p-2}}^{3p-2} + \|w_r\|_{B^{1+4\varepsilon}_{p,\infty}}^{\frac{3p-2}{3}},$$

note that F satisfies the assumptions of Lemma IV.15. Define also

$$\tau := \inf\{r \ge s, F(r) \le 1\} \wedge T$$

so that for any $r \in [s, \tau]$, $F(r) \ge 1$. If $\tau \ge t$, then applying Lemma IV.15, we obtain

$s = t_0 < \cdots < t_{N+1} = \tau$ such that $F(t_n) \lesssim (t_{n+1} - s)^{-\frac{3p-2}{2}}$. Let n be such that $t_n \leq t < t_{n+1}$, then using (IV.11)

$$\|w_t\|^{3p-2}_{L^{3p-2}} \lesssim 1 + F(t_n) \lesssim F(t_n) \lesssim (t-s)^{-\frac{3p-2}{2}}.$$

When $\tau < t$, note that the previous case ensure that

$$\|w_\tau\|^{3p-2}_{L^{3p-2}} \lesssim (\tau - s)^{-\frac{3p-2}{2}},$$

so that applying (IV.11) again

$$\|w_t\|^{3p-2}_{L^{3p-2}} \lesssim 1 + F(\tau) \lesssim (\tau - s)^{-\frac{3p-2}{2}}.$$

Taking $s = \frac{t}{2}$, we finally obtain

$$(1 \wedge \sqrt{t})\|w_t\|_{L^{3p-2}} \leq C \vee \tilde{K}_t \leq C \vee \tilde{K}_T.$$

In view of the blow-up criterion of Theorem IV.5, the fact that $L^{3p-2} \hookrightarrow \mathcal{C}^{-\varepsilon}$ for $p > \frac{4}{3\varepsilon}$ and the equivalence of the norms $\|v_t\|_{L^{3p-2}}$ and $\|w_t\|_{L^{3p-2}}$, this proves that solutions are indeed global.
The dependency of the upper-bound on the randomness of $\mathbf{z}$ comes from the fact that the $\mathbf{z}$ terms in $\tilde{K}_t$ only take into account what happens between the initial time and the current time, being 0 and t respectively. ▷

The fact that the bound on the solution to (IV.1) does not depend on the initial condition will be crucial in the following. We end this section with a bound on the moments of the solution, but first we prove the following lemma. From now on, denote by $u(t; u_0)$ the solution to (IV.1) at time t starting from u_0.

Lemma IV.17 – *For any* $u_0 \in \mathcal{C}^{-\varepsilon}$, $t \geq 0$ *and* $h > 0$,

$$u(t+h; u_0) = \stackrel{\bullet}{|}_{t,t+h} + \tilde{v}_{t,t+h}$$

where $\tilde{v}_{t,t+\cdot}$ *solves a shifted version of equation* (IV.6)

$$\tilde{v}_{t,t+h} = e^{-h\mathcal{H}} u(t; u_0) - \int_0^h e^{-(h-r)\mathcal{H}} \sum_{j=0}^{3} \tilde{v}^j_{t,t+r} \mathbf{z}^{(3-j)}_{t,t+r} \, dr.$$

Proof – From the mild formulation of equation (IV.6), we have for $h > 0$

$$\begin{aligned} v(t+h; u_0) &= e^{-(t+h)\mathcal{H}} u_0 - \int_0^{t+h} e^{-(t+h-r)\mathcal{H}} \sum_{j=0}^{3} v^j_r \mathbf{z}^{(3-j)}_r \, dr \\ &= e^{-h\mathcal{H}} \left(e^{-t\mathcal{H}} u_0 - \int_0^t e^{-(t-r)\mathcal{H}} \sum_{j=0}^{3} v^j_r \mathbf{z}^{(3-j)}_r \, dr \right) - \int_t^{t+h} e^{-(t+h-r)\mathcal{H}} \sum_{j=0}^{3} v^j_r \mathbf{z}^{(3-j)}_r \, dr \end{aligned}$$

$$= \Psi_{t,t+h} - \Psi_{0,t+h} + e^{-h\mathcal{H}}u(t+h;u_0) - \int_t^{t+h} e^{-(t+h-r)\mathcal{H}} \sum_{j=0}^{3} v_r^j \mathbf{z}_r^{(3-j)}\,dr,$$

so that $u(t+h;u_0) = \Psi_{0,t+h} + v(t+h;u_0) = \Psi_{t,t+h} + \tilde{v}_{t,t+h}$ where

$$\tilde{v}_{t,t+h} = e^{-h\mathcal{H}}u(t+h;u_0) - \int_0^{h} e^{-(h-r)\mathcal{H}} \sum_{j=0}^{3} v_{t+r}^j \mathbf{z}_{t+r}^{(3-j)}\,dr.$$

From the binomial expansion Lemma IV.2 and the relation between v and $\tilde{v}$ we get

$$\sum_{j=0}^{3} v_{t+r}^j \mathbf{z}_{t+r}^{(3-j)} = \sum_{j=0}^{3} \tilde{v}_{t,t+r}^j \mathbf{z}_{t,t+r}^{(3-j)}$$

and the expected expression of $\tilde{v}_{t,t+h}$ follows. ▷

Corollary IV.18 – *Let $u(\cdot;u_0) = \Psi_{0,\cdot} + v(\cdot;u_0)$ be the solution to equation (IV.1) starting from $u_0 \in \mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$ and v as in (IV.6), then for any $\varepsilon > 0$ small enough and p large enough*

$$\sup_{u_0\in\mathcal{C}^{\varepsilon}} \sup_{t\geq 0} \left(1 \wedge t^{\frac{3p-2}{2}}\right) \mathbb{E}\left[\|u(t;u_0)\|_{\mathcal{C}^{-\varepsilon}}^{3p-2}\right] < +\infty.$$

Proof – This is not a direct consequence of Proposition IV.16 as we cannot bound $\mathbb{E}[\tilde{K}_t]$ uniformly in $t \geq 0$. However, since the noise data $\mathbf{z}$ has the same law on intervals of the same size and the bound in Proposition IV.16 does not depend on the initial condition we are able to circumvent the issue. First note that

$$\sup_{u_0\in\mathcal{C}^{\varepsilon}} \sup_{0\leq t\leq 1} \left(1 \wedge t^{\frac{3p-2}{2}}\right) \mathbb{E}\left[\|u(t;u_0)\|_{\mathcal{C}^{-\varepsilon}}^{3p-2}\right] < +\infty$$

is indeed a direct consequence of Proposition IV.16. Now for the large times scenario, take $t > 1$, note that for $u_0 \in \mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$, $u(t;u_0) = \Psi_{t-1,t} + \tilde{v}_{t-1,t}$ where $\tilde{v}_{t-1,t-1+\cdot}$ is given by Lemma IV.17. Observe that $\tilde{v}_{t-1,t-1+\cdot}$ solves the same equation (IV.6) as v, but starting from $u(t-1;u_0)$ and driven by noise data $\mathbf{z}_{t-1,t-1+\cdot}$. From Proposition IV.16, $\|\tilde{v}_{t-1,t}\|_{L^{3p-2}}$ is bounded by some constant of the form

$$1 \wedge c\Big(\sup_{0\leq r\leq 1} \|\mathbf{z}_{t-1,t-1+\cdot}^{(1)}\|_{\mathcal{C}^{-\varepsilon}} + t^{\mathfrak{a}}\|\mathbf{z}_{t-1,t-1+\cdot}^{(2)}\|_{L^{\mathfrak{q}}\left([0,1],\mathcal{C}^{-\varepsilon}\right)} + t^{\mathfrak{a}}\|\mathbf{z}_{t-1,t-1+\cdot}^{(3)}\|_{L^{\mathfrak{q}}\left([0,1],\mathcal{C}^{-\varepsilon}\right)}\Big)\Big)^{\mathfrak{b}}.$$

Since from Lemma IV.1, the law of $\mathbf{z}_{t-1,t-1+\cdot}$ does not depend on t, one can take the expectation and get

$$\begin{aligned}\mathbb{E}\left[\|u(t;u_0)\|_{\mathcal{C}^{-\varepsilon}}^{3p-2}\right] &\lesssim \mathbb{E}\left[\|\Psi_{t-1,t}\|_{\mathcal{C}^{-\varepsilon}}^{3p-2}\right] + \mathbb{E}\left[\|\tilde{v}_{t-1,t}\|_{\mathcal{C}^{-\varepsilon}}^{3p-2}\right] \\ &\lesssim \mathbb{E}\left[\|\Psi_{t-1,t}\|_{\mathcal{C}^{-\varepsilon}}^{3p-2}\right] + \mathbb{E}\left[\|\tilde{v}_{t-1,t}\|_{L^{3p-2}}^{3p-2}\right]\end{aligned}$$

where the right-hand side is bounded uniformly both in t and u_0. ▷

IV.3 – The Anderson Φ^4_2 measure

The previous construction ensures that Equation (IV.1)

$$\begin{cases} \partial_t u + \mathcal{H}u + u^{\diamond 3} = \sqrt{2}\zeta, \\ u_{t=0} = u_0, \end{cases}$$

is globally well-posed in $\mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$. We turn in this section to the properties of $t \mapsto u(t;\cdot)$ as a Markov process on $\mathcal{C}^{-\varepsilon}$ and its transition semigroup $(P_t)_{t\geq 0}$ defined for $t \geq 0$ and $\Phi \in C_b\big(\mathcal{C}^{-\varepsilon}\big)$ as

$$P_t\Phi(u_0) := \mathbb{E}\left[\Phi\Big(u(t;u_0)\Big)\right]$$

for any initial data $u_0 \in \mathcal{C}^{-\varepsilon}$.

IV.3.1 The transition semi-group and Feller property

As we intend to prove Markov property for $t \mapsto u(t;\cdot)$, we give a more precise description of $\tilde{v}$ defined in Lemma IV.17. For $h \geq 0$, $v_0 \in \mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$ and noise data vector $\mathbf{y}$ having the same regularity properties as $\mathbf{z}$ we write $v(h;v_0;\mathbf{y})$ the solution to (IV.6) at time h, starting from v_0 and driven by $\mathbf{y}$. This notation will simplify the proof of the following proposition.

Proposition IV.19 – *Let $u_0 \in \mathcal{C}^{-\varepsilon}$, $t,h \geq 0$. Then for any $\Phi \in C_b\big(\mathcal{C}^{-\varepsilon}\big)$*

$$\mathbb{E}\left[\Phi\Big(u(t+h;u_0)\Big)\Big|\mathcal{F}_t\right] = P_t\Phi\Big(u(t;u_0)\Big).$$

Note that this proves that $t \mapsto P_t$ is indeed a semigroup, and the Markov property of $t \mapsto u(t;\cdot)$ is then a direct consequence of this proposition.

Proof – In view of Lemma IV.17, $u(t+h;u_0) = \overset{\bullet}{|}_{t,t+h} + v\Big(h;u(t;u_0);\mathbf{z}_{t,t+\cdot}\Big)$ where $u(t;u_0)$ is $\mathcal{F}_t$-measurable while $\mathbf{z}_{t,t+\cdot}$ is $\mathcal{F}_t$-independent (see Lemma IV.1). Thus, according to proposition 1.12 in [DZ14],

$$\begin{aligned} \mathbb{E}\left[\Phi\Big(u(t+h;u_0)\Big)\Big|\mathcal{F}_t\right] &= \mathbb{E}\left[\Phi\Big(\overset{\bullet}{|}_{t,t+h} + v\Big(h;u(t;u_0);\mathbf{z}_{t,t+\cdot}\Big)\Big)\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\Phi\Big(\overset{\bullet}{|}_{t,t+h} + v\Big(h;u(t;u_0);\mathbf{z}_{t,t+\cdot}\Big)\Big)\right]. \end{aligned}$$

Moreover, since $\mathbf{z}_{t,t+\cdot}$ and $\mathbf{z}$ have the same law (see again Lemma IV.1), for any $v_0 \in \mathcal{C}^{-\varepsilon}$ we get

$$\begin{aligned} \mathbb{E}\left[\Phi\Big(\overset{\bullet}{|}_{t,t+h} + v\Big(h;v_0;\mathbf{z}_{t,t+\cdot}\Big)\Big)\right] &= \mathbb{E}\left[\Phi\Big(\overset{\bullet}{|}_{0,h} + v\Big(h;v_0;\mathbf{z}\Big)\Big)\right] \\ &= \mathbb{E}\left[\Phi\Big(u(h;v_0)\Big)\right] \end{aligned}$$

$$= P_h\Phi\Big(v_0\Big),$$

Applying this in $v_0 = u(t;u_0)$ and we finally get

$$P_h\Phi\Big(u(t;u_0)\Big) = \mathbb{E}\left[\Phi\Big(u(t+h;u_0)\Big)\Big|\mathcal{F}_t\right].$$

$\triangleright$

We make the following observation: in Theorem IV.5, not only do we obtain local existence, but also that the solution v to (IV.6) depends continuously on the initial data in $\mathcal{C}^{-\varepsilon}$. As a direct consequence, we obtain Feller property for the semigroup $(P_t)_{t\geq 0}$.

Corollary IV.20 – *The semigroup $(P_t)_{t\geq 0}$ has the Feller property, that is, for any $\Phi \in C_b\Big(\mathcal{C}^{-\varepsilon}\Big)$ and $t \geq 0$ then $P_t\Phi \in C_b\Big(\mathcal{C}^{-\varepsilon}\Big)$.*

We conclude this subsection with the existence of invariant measures for the semigroup $(P_t)_{t\geq 0}$. Do note that all the above remain true if we were to tune ε to be even smaller, for the purpose of the next result, introduce $\varepsilon_0 = \frac{\varepsilon}{2}$ and see $(P_t)_{t\geq 0}$ as acting on $C_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$. To that end we prove an ergodic theorem for the dual semigroup $(P^*_t)_{t\geq 0}$ of $(P_t)_{t\geq 0}$ acting on the set of probability measures on $\mathcal{C}^{-\varepsilon_0}$. Note we use a compactness argument so that we cannot ensure uniqueness of an invariant measure at this stage.

Proposition IV.21 – *Let $u_0 \in \mathcal{C}^{-\varepsilon_0}(\mathbb{T}^2)$, there is a sequence of times $0 < t_0 < t_1 < \cdots \to +\infty$ and a probability measure μ_{u_0} on $\mathcal{C}^{-\varepsilon_0}(\mathbb{T}^2)$ such that*

$$\frac{1}{t_k}\int_0^{t_k} P^*_r\delta_{u_0}\, dr \to \mu_{u_0}$$

where the convergence holds weakly. As such, μ_{u_0} is an invariant measure for $(P_t)_{t\geq 0}$ acting on $C_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$.

Proof – We intend to use Krylov-Bogoliubov theorem as given in theorem 3.1.1 of [DZ96], set $R_t = \frac{1}{t}\int_0^t P^*_r\delta_{u_0}\, dr$, it is enough to prove that the sequence $(R_{t_k})_{k\in\mathbb{N}}$ is uniformly tight as a sequence of probability measures on $\mathcal{C}^{-\varepsilon_0}$, for some $0 < t_0 < t_1 < \cdots \to +\infty$. In particular we are not bothered by the behavior of R_t close to 0, only when $t \to +\infty$. Let $p > 1$ be as in Proposition IV.18, $t \geq 0$ and $R > 0$. We begin with the following observation that arises from the very definition of P^*. Using Markov and Jensen's inequality

$$\begin{aligned} R_t\left(\{u_0 \in \mathcal{C}^{-\varepsilon},\ \|u_0\|_{\mathcal{C}^{-\varepsilon}} > R\}\right) &= \frac{1}{t}\int_0^t \mathbb{P}\left(\|u(r;u_0)\|_{\mathcal{C}^{-\varepsilon}} > R\right) dr \\ &\lesssim \frac{1}{Rt}\int_0^t \mathbb{E}\left[\|u(r;u_0)\|^{3p-2}_{\mathcal{C}^{-\varepsilon}}\right]^{\frac{1}{3p-2}} dr. \end{aligned}$$

Then Proposition IV.18 yields

$$R_t\left(\{u_0 \in \mathcal{C}^{-\varepsilon},\ \|u_0\|_{\mathcal{C}^{-\varepsilon}} > R\}\right) \lesssim \frac{1}{Rt}(1+t).$$

As $\varepsilon > \varepsilon_0$, $\{u_0 \in \mathcal{C}^{-\varepsilon},\ \|u_0\|_{\mathcal{C}^{-\varepsilon}} \leq R\}$ is a compact subset of $\mathcal{C}^{-\varepsilon_0}$ and given that for any $t_0 > 0$, $\frac{1+t}{t}$ is bounded uniformly in t, there exists a sequence $0 < t_0 < t_1 < \cdots \to +\infty$ such that $(R_{t_k})_{k\in\mathbb{N}}$ is tight. As such, up to extraction (which we still denote by $0 < t_0 < t_1 < \cdots \to +\infty$), there exists a limiting measure μ_{u_0} on $\mathcal{C}^{-\varepsilon_0}$ such that R_{t_k} converges weakly to μ_{u_0}. Theorem 3.1.1 in [DZ96] then ensures that such a μ_{u_0} is an invariant measure to the semigroup $(P_t)_{t\geq 0}$ acting on $C_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$. ▷

IV.3.2 Perspective: Strong Feller property and ergodicity

This final subsection aims at giving the idea behind the work in progress on which is based this whole chapter, the final goal being to enhance Chapter III by providing a constrction of the measure through the dynamic, uniqueness of the invariant measure constructed therein. So far, we proved that the dynamic satisfying (IV.1) (has a transition semigroup that) is Markovian and satisfies the Feller property, as such it admits an invariant measure. We do not go into much technical details in this subsection and rather give a schematic and concise overview of the natural continuation of this Chapter. Rather than mapping $C_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$ to itslef, the strong Feller property consists in P_t mapping the set of bounded measurable functions $B_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$ to $C_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$ for positive t. In [TW18], they are able to prove that their transition semigroup has some low index Hölder regularity, which is enough. However, in [DD20], their argument allows to obtain local Lipschitz bounds in the sense that, if $\Phi \in B_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$, then $P_t\Phi$ is locally Lipschitz on $\mathcal{C}^{-\varepsilon_0}$. The road map to adapt this strategy is described here, putting aside the rigorous mathematical difficulty to emphasize the strategy. Consider the Anderson Φ^4_2 equation on $\mathbb{T}^2$ (IV.1)

$$\begin{cases} \partial_t u + \mathcal{H}u + u^{\diamond 3} = \sqrt{2}\zeta, \\ u_{t=0} = u_0, \end{cases}$$

for which we know there exists a unique global solution in $\mathcal{C}^{-\varepsilon_0}$. Differentiating with respect to the initial condition u_0, we write $\eta^h := d_{u_0}u\cdot h$ for the differential of u with respect to u_0 in the direction $h \in \mathcal{C}^{-\varepsilon_0}$. Then η^h satisfies the linearized equation

$$\begin{cases} \partial_t \eta^h + \mathcal{H}\eta^h + 3u^{\diamond 2}\eta^h = 0, \\ \eta^h_{t=0} = h, \end{cases} \tag{IV.12}$$

(this, of course, does not make sense as it is, and requires a truncation of the noise or the non-linearity to be made rigorous). As (IV.12) is a linear equation, we can make extensive use of the regularization properties of the Anderson heat semigroup $e^{-t\mathcal{H}}$ together with Grönwall's lemma to obtain estimates under the form

$$\|\eta^h_t\|_{\mathcal{C}^{\varepsilon_0+\kappa}} \lesssim t^{-\frac{2\varepsilon_0+\kappa}{2}} \exp\left(c\int_0^t V\Big(u(s;u_0)\Big)\,ds\right)$$

where $\kappa > 0$ is small enough, $u_0 \in \mathcal{C}^{-\varepsilon_0}$ and V is a non-negative non-linear functional. Note that the exponential term on the right-hand side may blow-up as t goes to $+\infty$. This is an issue as the second main part of the proof is to use a Bismut-Elworthy-Li formula for the semigroup $(P_t)_{t\geq 0}$ such as the one in [DZ97, Theorem 2.1] (once again, this also requires to work with a finite dimensional system to make sure all the assumptions needed are satisfied). To circumvent the exponential blow-up, the idea of [DD20] is to apply this formula to a modified semigroup given for $t \geq 0$ and $\Phi \in C_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$ by

$$S_t\Phi(u_0) := \mathbb{E}\left[e^{-c\int_0^t V(u(s;u_0))\,ds}\Phi\Big(u(t;u_0)\Big)\right].$$

This new semigroup $(S_t)_{t\geq 0}$ satisfies a similar Kolmogorov equation as $(P_t)_{t\geq 0}$, but with an extra term due to the potential V, they are linked by the mild equation

$$P_t\Phi(u_0) = S_t\Phi(u_0) + c\int_0^t S_{t-s}\Big(VP_t\Phi\Big)(u_0)\,ds.$$

This type of perturbative argument is not unfamiliar as it was the key to obtain Schauder estimates for $\mathcal{H}$ (see Equation (I.5)). Obtaining good enough estimates on $(S_t)_{t\geq 0}$ using the Bismut-Elowrthy-Li formula will therefore yield estimates on $(P_t)_{t\geq 0}$ and in the end, we are able to get an estimate on the gradient of $P_t\Phi$ for $\Phi \in B_b\Big(\mathcal{C}^{-\varepsilon_0}\Big)$

$$|dP_t\Phi(u_0)\cdot h| \lesssim c_t\|h\|_{\mathcal{C}^{-\varepsilon_0}} \sup_{v\in\mathcal{C}^{-\varepsilon_0}} \|\Phi(v)\|(1+\|u_0\|_{\mathcal{C}^{-\varepsilon_0}})^\gamma$$

which in turn, yields local Lipschitz estimates on $P_t\Phi$. While this ensures that the semigroup satisfies the strong Feller property, note that it is a strictly stronger statement in that it also gives a quantitative bound on the derivative. Precise estimates and technical details of course are missing from this whole discussion, a proper truncation of (IV.1) has to be made as well as keeping track of the truncation parameter when doing the estimates to ensure the whole argument can be passed to the limit. One of the issues when dealing with equations driven by the singular operator $\mathcal{H}$ being that usual truncations based on Δ do not commute with $\mathcal{H}$, hence the need of a truncation specifically designed for $\mathcal{H}$. A spectral projection has been defined in Subsection I.2.3 and used to prove measure invariance in Chapter III. Another issue comes from the singular nature of $\mathcal{H}$. The Anderson heat semigroup $e^{-t\mathcal{H}}$ does not provide better regularization effect than $\mathcal{C}^{1^-}$, this is why we had to work at the level of the remainder term w in Section IV.2. While it requires precise analysis, the paracontrolled expansion captures the singular aspect of the Anderson operator, allows to use the usual tools and is the key to carry on the procedure we just described.
The final step in proving ergodicity would be to establish a support theorem for the law $\mathbb{P}_{u(\cdot;u_0)}$ of u starting from an initial data $u_0 \in \mathcal{C}^{-\varepsilon_0}$. While this has been done in the usual Φ_2^4 case in [TW18, Section 6], a similar result exists in the case of the parabolic Anderson model in [CF18]. Combining both approaches, we should be able to state a similar support theorem in our context of Anderson Φ_2^4 equation and finally obtain exponential ergodicity of the dynamic, proving both uniqueness of the invariant measure, as well as convergence at an exponential rate towards this measure.

V Non-local quasilinear PAM

Contents

V.1 – Introduction

We are interested in this work in giving a solution theory for some quasilinear singular stochastic partial differential equations

$$\partial_t u - A(f(u))\Delta u = B(g(u))\,\xi \tag{V.1}$$

whose coefficients $A(f(u)), B(f(u))$ are *nonlocal functionals* of some functions of the solution u. We do so here in the mildly singular setting of the 2-dimensional torus $\mathbb{T}^2$, for a space white noise ξ. We need some notations to set the stage of our main result, Theorem V.1 below.

Denote by (e_1, e_2) the canonical basis of $\mathbb{Z}^2 \subset \mathbb{R}^2$, and for $i \in \{1,2\}$ and a function $\sigma : \mathbb{T}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ set

$$(D_i\sigma)(x,k) := \sigma(x, k+e_i) - \sigma(x,k),$$

and $\mathbf{D} := (D_1, D_2)$. Write $\mathcal{F}$ for the Fourier transform on $\mathbb{T}^2$ and denote by $\widehat{\sigma}(n,k) = (\mathcal{F}\sigma)(n,k)$ the Fourier transform of the function $\sigma(\cdot,k)$ for each $k \in \mathbb{Z}^2$. Given $s \in \mathbb{R}$ and $\alpha > 0$ we define the class Σ^s_α of functions $\sigma : \mathbb{T}^2 \times \mathbb{Z}^2 \to \mathbb{C}$ such that

$$\left\| \partial_x^i \mathbf{D}^j \sigma(\cdot,k) \right\|_{\mathcal{C}^\alpha} \lesssim (1+|k|)^{s-|j|} \tag{V.2}$$

for all multi-indices i, j in $\mathbb{N}^2$, and for which there exists a constant $0 < \mu < 1$ such that

$$(\mathcal{F}\sigma)(n,k) = 0, \quad \text{whenever } |n| > \mu(1+|k|). \tag{V.3}$$

For such a symbol, we can easily bound $\widehat{\sigma}$ uniformly in (n,k) by

$$\left|\widehat{\sigma}(n,k)\right| \lesssim \langle n\rangle^{-4} \langle k\rangle^{s}\, \mathbf{1}_{|n|\le\mu(1+|k|)}. \tag{V.4}$$

For a and b in Σ^s_α we denote by A and B the pseudodifferential operators with symbols a and b respectively. The convolution operator with a function $\mathcal{F}^{-1}(\sigma)$ on $\mathbb{T}^2$ whose Fourier transform $\sigma(k)$ satisfies (V.2) provides an elementary example of a function $\sigma \in \Sigma^0_\alpha$ – here it is independent of x and σ is supported on the diagonal $\{n = k\}$. Similarly, ad hoc conditions on a kernel K ensure that the integral operator $v \mapsto \int K(\cdot, y)v(y)dy$ is associated with a symbol $\sigma \in \Sigma^s_\alpha$ for some s. A negative s means an operator that regularizes.

We say that A *preserves positivity* if $A(v) > 0$ whenever v is bounded below by a positive constant. Positivity preserving pseudodifferential operators with a symbol in Σ^0_α for some $\alpha > 0$ are given by some kernels $\mu(x, \cdot)$ that are finite non-negative measures for all $x \in \mathbb{T}^2$ such that testing $A(v)$ against a smooth test function ℓ on $\mathbb{T}^2$ one has

$$A(v)(\ell) = \iint v(y)\mu(x, dy)\ell(x)dx. \tag{V.5}$$

This is the case of the preceding integral operator when K is bounded below by a positive constant. In the renormalized equation (V.6) below we regularize the space white noise ξ into a smooth function

$$\xi^\varepsilon := \mathcal{F}^{-1}\Big(\chi(\varepsilon|\cdot|)\,\mathcal{F}(\xi)(\cdot)\Big)$$

using a 'smooth Fourier cut-off' $\chi : [0, \infty) \to [0, 1]$ of class C^∞, equal to 1 on $[0, 1]$ and 0 on $[2, \infty)$.

Theorem V.1 – *Pick two regularity exponents $2/3 < \beta < \alpha < 1$ and $s \leq 0$. Take two symbols $a, b \in \Sigma^s_\alpha$ and assume that the operator A preserves positivity. Take further $f, g \in C^3_b(\mathbb{R}, \mathbb{R})$.*

(a) There exists two deterministic diverging functions $c_a, c_b : (0, 1] \to C^\infty(\mathbb{T}^2)$ and a positive random time T such that the solution u^ε to the equation

$$\partial_t u^\varepsilon - A\Big(f(u^\varepsilon)\Big)\Delta u^\varepsilon = B\Big(g(u^\varepsilon)\Big)\,\xi^\varepsilon + c_a(\varepsilon)\left(\frac{B\Big(g(u^\varepsilon)\Big)}{A\Big(f(u^\varepsilon)\Big)}\right)^2 f'(u^\varepsilon) - c_b(\varepsilon)\frac{B\Big(g(u^\varepsilon)\Big)}{A\Big(f(u^\varepsilon)\Big)}\,g'(u^\varepsilon) \tag{V.6}$$

with initial condition $u_0 \in \mathcal{C}^\alpha(\mathbb{T}^2)$ converges in probability in the parabolic α-Hölder space on $[0, T] \times \mathbb{T}^2$, as $\varepsilon \in (0, 1]$ goes to 0.

(b) The singular PDE

$$\partial_t u - A(f(u))\Delta u = B(g(u))\,\xi$$

has a well-defined formulation in a space of paracontrolled functions, where it has a unique solution (u, u'). The limit of the u^ε is given by u.

This statement gives back the result first proved in [OW19; FG19; BDH16] by Otto & Weber, Furlan & Gubinelli and Bailleul, Debussche & Hofmanová when A and B are the identity operators. The approach of Otto & Weber was later fully developed in [OSSW23; LOT23; LOTT22] to provide a variant of regularity structures well-adapted to a certain class of quasilinear singular stochastic PDEs. The structural assumption from which they get back their counterterm naturally leads to a counterterm that is a local functional of u^ε. It would be interesting to see whether their approach can be twicked to give back our result where the counterterm is nonlocal. The approach of [BDH16] was developed in a paracontrolled setting in Bailleul & Mouzard's work [BM23] and fully developped in a regularity structure setting in Bailleul, Hoshino & Kusuoka's work [BHK23].

In the proof of Theorem V.1 we obtain point (a) as a corollary of point (b). The proof of point (b) involves two different tasks.

– Build an analytical setting where to formulate the equation as a well-defined fixed point problem.
– Construct a random variable that plays the role of an ill-defined polynomial functional of the noise. It is required as an independent ingredient in the analytical setting.

We formulate the equation as a fixed point in a space of paracontrolled functions in Section V.2 . We prove therein that it is well-posed locally in time. The construction of the polynomial functional of the noise is done in Section V.3.

Notations – *Throughout, for $\alpha \in (0,1)$ and a finite positive time horizon T we set*

$$\mathscr{C}_T^\alpha := C\Big([0,T], \mathcal{C}^\alpha(\mathbb{T}^2)\Big) \cap C^{\alpha/2}\Big([0,T], L^\infty(\mathbb{T}^2)\Big).$$

This space coincides with the α-Hölder parabolic space, with an equivalent norm. We will denote throughout by A and B the pseudodifferential operators associated with the symbols a and b respectively.

V.2 – Paracontrolled formulation of the equation

The setting of paracontrolled calculus is now familiar enough that we can go straight to the point. We refer the reader to Gubinelli & Perkowski's lecture notes [GP17] for an introduction. We refer to the original article [GIP15] of Gubinelli, Imkeller & Perkowski or the advanced work [BB19] of Bailleul & Bernicot to see paracontrolled calculus in action in some situations that are more involved than what we need here.

In its basic form a paracontrolled structure is a Banach space of functions of the form

$$v = \mathsf{P}_{v'}X + v^\sharp \tag{V.7}$$

where $v' \in \mathcal{C}^{\alpha_1}(\mathbb{T}^2)$ with $\alpha_1 > 0$, and $X \in \mathcal{C}^\alpha(\mathbb{T}^2)$, and $v^\sharp \in \mathcal{C}^{\alpha_2}(\mathbb{T}^2)$ with $\alpha_2 > \alpha$. Given that $\mathsf{P}_{v'}X \in \mathcal{C}^\alpha(\mathbb{T}^2)$ from the continuity estimate of Proposition I.1 in Appendix A.3, one sees $v^\sharp$ as a 'remainder' term in the above decomposition of v. This paracontrolled structure is stable by nonlinear functions, a consequence of Bony's paralinearization result, Proposition I.4 in Chapter I. In the study of the semilinear version of Equation (V.1), where $A = 1$ say, we can formulate the equation as a fixed point problem in some space of time-dependent paracontrolled functions provided that we have an appropriate definition of the product $X\xi$. An explicit choice of X as a function of ξ brings back the question of that definition to a problem about random variables only. Once this probability problem is understood solving the equation itself is a purely deterministic problem in a given (ω-dependent) Banach space of paracontrolled functions (where ω stands for the chance element). We will follow here a similar strategy, following Bailleul, Debussche & Hofmanová's reformulation of the quasilinear equation (V.1) as a 'perturbation' of a semilinear equation [BDH16]

$$\partial_t u - A(f(u_0^T))\Delta u = B(g(u))\xi + D_0(u)\Delta u, \tag{V.8}$$

where

$$u_0^T := e^{T\Delta}(u_0) \in C^\infty(\mathbb{T}^2).$$

and

$$D_0(u) := A(f(u)) - A(f(u_0^T))$$

In order to proceed like that we need first to make sure that both $A(f(v))$ and $B(g(v))$ have a paracontrolled structure if v does and that this structure is preserved by the equation.

V.2.1 Stability of the paracontrolled structure

Recall the notation of the modified paraproduct $\overline{\mathsf{P}}$ from Appendix A.3. We now fix X for the remainder of this chapter

$$X = -\Delta^{-1}\xi,$$

with null mean on $\mathbb{T}^2$. Pick

$$\frac{2}{3} < \beta < \alpha < 1$$

and define the space $\mathbf{C}^\beta_{\alpha,T}(X)$ as the set of all functions $(u, u') \in \mathscr{C}^\alpha_T \times \mathscr{C}^\beta_T$ such that

$$u^\sharp := \left(u - \overline{\mathsf{P}}_{u'}X\right) \in \mathscr{C}^\alpha_T \qquad \text{and} \qquad \sup_{0<t\leq T} t^{\frac{2\beta-\alpha}{2}} \|u^\sharp(t)\|_{\mathcal{C}^{2\beta}} < +\infty \tag{V.9}$$

which we equip with the norm

$$\|(u,u')\|_{\mathbf{C}^\beta_{\alpha,T}(X)} := \|u'\|_{\mathscr{C}^\beta_T} + \|u^\sharp\|_{\mathscr{C}^\alpha_T} + \sup_{0<t\leq T} t^{\frac{2\beta-\alpha}{2}} \|u^\sharp(t)\|_{\mathcal{C}^{2\beta}}.$$

We will look for a solution to Equation (V.1), equivalently Equation (V.8), as an element of $\mathbf{C}^\beta_{\alpha,T}(X)$. The reformulation (V.8) of Equation (V.1) involves a number of operations. We need to check how the a priori paracontrolled structure of a solution to (V.8) interacts with these operations.

a) Pseudodifferential operators and paraproducts. The class of symbols Σ^s_α is defined from the constraints (V.2) and (V.3). For $\alpha_1 > 0$ and $h_1 \in \mathcal{C}^{\alpha_1}(\mathbb{T}^2)$ we check that the operator $\mathsf{P}_{h_1}(\cdot)$ has a symbol in the class $\Sigma^0_{\alpha_1}$. We read on the condition (V.2) that the class of operators associated to Σ^s_α is a subclass of the set of pseudodifferential operators Ψ^s_{11}. The operators in this class send continuously any $\mathcal{C}^\gamma(\mathbb{T}^2)$ space into $\mathcal{C}^{\gamma-s}(\mathbb{T}^2)$ if $\gamma > 0$ and $(\gamma - s) > 0$ are non-integers – this is the classical Schauder estimate.

One gets as follows the structure (V.5) of the operators with a symbol in Σ^0_α that are positivity preserving. For $v \in \mathcal{C}^\gamma(\mathbb{T}^2)$ with γ one has from the classical Schauder estimate $A(v) \in \mathcal{C}^\gamma(\mathbb{T}^2)$. Now for $v \in \mathcal{C}^\gamma(\mathbb{T}^2)$, since $2\|v\|_\infty \pm v \geq \|v\|_\infty > 0$ the positivity preserving property yields $A(2\|v\|_\infty \pm v) \geq 0$, that is $\pm A(v) \leq 2\|v\|_\infty A(\mathbf{1})$, i.e. $|A(v)| \leq 2\|v\|_\infty A(\mathbf{1})$. For any $x \in \mathbb{T}^2$ the linear form $A(\cdot)(x)$ on $\mathcal{C}^\gamma(\mathbb{T}^2)$ thus has a continuous extension into a linear form on $C^0(\mathbb{T}^2)$. Riesz representation theorem then gives the existence of a non-negative measure $\mu(x,\cdot)$ such that $A(\cdot)(x)$ is the integration map against the measure $\mu(x,\cdot)$.

To check that the paracontrolled structure (V.7) is stable by the nonlocal maps A associated with some symbols $\sigma \in \Sigma^s_{\alpha_1}$ we can use the following commutator estimate from Proposition 3.8 in Bailleul, Dang, Ferdinand & Tô's work [BDFT23b], together with the classical Schauder estimate.

Proposition V.2 – *Pick $s \leq 0, \alpha_1 \in (0,1)$ and $\alpha_2 > 0$. For any $\sigma \in \Sigma^s_{\alpha_1}$ and $h_1 \in \mathcal{C}^{\alpha_1}(\mathbb{T}^2)$, for any $h_2 \in \mathcal{C}^{\alpha_2}(\mathbb{T}^2)$ one has*

$$\Big\| Op(\sigma)(\mathsf{P}_{h_1} h_2) - \mathsf{P}_{h_1}\Big(Op(\sigma)(h_2)\Big)\Big\|_{\mathcal{C}^{\alpha_1+\alpha_2-s}} \lesssim_a \|h_1\|_{\mathcal{C}^{\alpha_1}} \|h_2\|_{\mathcal{C}^{\alpha_2}}.$$

b) Structural elements in the right hand side of* (V.8)*. From Proposition I.4 on paralinearization, Proposition I.3 on iterated paraproducts and the continuity result for the corrector in Proposition I.2, one can expand the first ill-defined product

$$B(f(u))\,\xi = \mathsf{P}_{B(f(u))}\xi + \mathsf{P}_\xi B(f(u)) + u'g'(u)\Pi(B(X),\xi) + g'(u)\Pi(B(u^\sharp),\xi) + \sharp_B \tag{V.10}$$

where the ill-defined product $\Pi(B(X),\xi)$ is to be understood as its renormalized counterpart defined in Section V.3 (note that $u'g'(u)\Pi(B(X),\xi)$ is the only term featuring this ill-defined product). The term

$$\begin{aligned}\sharp_B = \mathsf{C}(u'g'(u),B(X),\xi) + \mathsf{C}(g'(u),u^\sharp,\xi) + \Pi\Big([B,\mathsf{P}_{u'g'(u)}](X)\,,\,\xi\Big) + \Pi\Big([B,\mathsf{P}_{g'(u)}](u^\sharp)\,,\,\xi\Big) \\ + \Pi\Big(B\Big(\mathsf{P}_{g'(u)}\overline{\mathsf{P}}_{u'}X - \mathsf{P}_{u'g'(u)}X\Big)\,,\,\xi\Big) + \Pi\Big(B\Big(g(u) - \mathsf{P}_{g'(u)}u\Big),\xi\Big)\end{aligned}$$

denotes an element of the space $C\big(\big[0,T\big],\mathcal{C}^{2\alpha+\beta-2}(\mathbb{T}^2)\big)$ that is polynomial in (u,u') and depends continuously on ξ. Given $\gamma \in \mathbb{R}$ write $=_\gamma$ to mean equality up to an element of $C\Big([0,T],\mathcal{C}^\gamma(\mathbb{T}^2)\Big)$. One also has for the Laplace term

$$\begin{aligned}D_0(u)\Delta u &= D_0(u)\Big(\Delta\overline{\mathsf{P}}_{u'}X + \Delta u^\sharp\Big) \\ &= D_0(u)\Big(-\overline{\mathsf{P}}_{u'}\xi + [\Delta,\overline{\mathsf{P}}_{u'}]X + \Delta u^\sharp\Big) \\ &=_{\alpha+\beta-2} D_0(u)\Big(-\overline{\mathsf{P}}_{u'}\xi + \Delta u^\sharp\Big) \\ &=_{\alpha+\beta-2} -\mathsf{P}_{D_0(u)}\Big(\overline{\mathsf{P}}_{u'}\xi\Big) - \Pi\Big(D_0(u),\overline{\mathsf{P}}_{u'}\xi\Big) + D_0(u)\Delta u^\sharp \\ &=_{\alpha+\beta-2} -\mathsf{P}_{D_0(u)u'}\xi - \Pi\Big(A(f(u)),\overline{\mathsf{P}}_{u'}\xi\Big) + D_0(u)\Delta u^\sharp \\ &=_{\alpha+\beta-2} -\mathsf{P}_{D_0(u)u'}\xi - u'\Pi\Big(A(f(u)),\xi\Big) + D_0(u)\Delta u^\sharp,\end{aligned}$$

where we used successively the paracontrolled expansion of u, the commutation lemma

A.10 to get rid of the $[\Delta, \overline{\mathsf{P}}_{u'}]$ term and switch back and forth between $\overline{\mathsf{P}}$ and P, the commutation property I.3 to merge the paraproducts, the fact that, as $D_0(u) = A(f(u)) - A(f(u_0^T))$ with u_0^T smooth, so is $A(f(u_0^T))$, and the $\Pi\Big(A(f(u_0^T)), \overline{\mathsf{P}}_{u'}\xi\Big)$ term has $C_T\mathcal{C}^{\alpha+\beta-2}$ regularity, and finally the corrector estimate I.2 for the remaining resonant term. The resonant product in the last step can be expanded as follows using the paracontrolled expansion of u and paralinearization lemma I.5

$$\Pi\Big(A(f(u)), \xi\Big) =_{2\alpha+\beta-2} u'f'(u)\Pi\Big(A(X), \xi\Big) + f'(u)\Pi\Big(A(u^\sharp), \xi\Big). \tag{V.11}$$

Note that $-(u')^2 f'(u)\Pi\Big(A(X), \xi\Big)$ is the only term featuring the renormalized product $\Pi\Big(A(X), \xi\Big)$. This gives

$$\begin{aligned} D_0(u)\Delta u = &-\mathsf{P}_{D_0(u)u'}\xi - u'^2 f'(u)\Pi\Big(A(X), \xi\Big) \\ &+ u'f'(u)\Pi\Big(A(u^\sharp), \xi\Big) + D_0(u)\Delta u^\sharp + \sharp_A \end{aligned}$$

and

$$\begin{aligned} \sharp_A = &\, D_0(u)\,[\Delta, \overline{\mathsf{P}}_{u'}](X) + \Big(\mathsf{P}_{D_0(u)u'}\xi - D_0(u)\,\overline{\mathsf{P}}_{u'}X\Big) - \mathsf{P}_{\overline{\mathsf{P}}_{u'}\xi}D_0(u) + \Pi\Big(\overline{\mathsf{P}}_{u'}\xi, A(f(u_0^T))\Big) \\ &- \Big\{\Pi\Big(\overline{\mathsf{P}}_{u'}\xi, A(f(u))\Big) - u'\Pi\Big(\xi, A(f(u))\Big)\Big\} - u'\,\mathsf{C}\Big(u'f'(u), A(X), \xi\Big) + u'\,\mathsf{C}\Big(f'(u), u^\sharp, \xi\Big) \\ &+ u'\,\Pi\Big([A, \mathsf{P}_{u'f'(u)}](X), \xi\Big) + u'\,\Pi\Big([A, \mathsf{P}_{f'(u)}](u^\sharp), \xi\Big) \\ &+ u'\,\Pi\Big(A\Big(\mathsf{P}_{f'(u)}\overline{\mathsf{P}}_{u'}X - \mathsf{P}_{u'f'(u)}X\Big), \xi\Big) + u'\,\Pi\Big(A\Big(f(u) - \mathsf{P}_{f'(u)}u\Big), \xi\Big) \end{aligned}$$

is an element of $C\Big([0,T], \mathcal{C}^{2\alpha+\beta-2}(\mathbb{T}^2)\Big)$ that is polynomial and continuous in (u, u') and continuous in ξ. Note that although $\Delta u^\sharp \in \mathcal{C}^{2\beta-2}(\mathbb{T}^2)$ at every positive time t it cannot be considered as a remainder term as its $\mathcal{C}^{2\beta-2}$-norm blows up when $t > 0$ goes to 0. Using Lemma A.13 we can then rewrite Equation (V.8) as a system of coupled equation

$$\begin{aligned} \partial_t u - A(f(u_0^T))\Delta u = &\, \mathsf{P}_{B(g(u)) - A(f(u))u'}\xi + \Big\{A(f(u)) - A(f(u_0^T))\Big\}\Delta u^\sharp \\ &+ g'(u)\Pi\Big(B(u^\sharp), \xi\Big) - u'f'(u)\,\Pi\Big(A(u^\sharp), \xi\Big) + \sharp_1(u, u') \end{aligned}$$

and

$$\begin{aligned} \partial_t u^\sharp - A(f(u_0^T))\Delta u^\sharp = &\, \mathsf{P}_{B(g(u)) - A(f(u))u'}\xi + \Big\{A(f(u)) - A(f(u_0^T))\Big\}\Delta u^\sharp \\ &+ g'(u)\Pi\Big(B(u^\sharp), \xi\Big) - u'f'(u)\,\Pi\Big(A(u^\sharp), \xi\Big) + \sharp_2(u, u') \end{aligned}$$

where $\sharp_1(u, u')$ and $\sharp_2(u, u')$ are elements of $C\Big([0,T], \mathcal{C}^{\alpha+\beta-2}(\mathbb{T}^2)\Big)$ that are non-linear functions of u, multilinear in $(u', u^\sharp)$ and depend continuously on the enhanced noise data

$$\Big(\xi, \Pi(A(X), \xi), \Pi(B(X), \xi)\Big).$$

Note that in Equations (V.10) and (V.11), we emphasized the dependency on the renormalized products $\Pi(B(X), \xi)$ and $\Pi(A(X), \xi)$ respectively. This will be useful when we investigate the consistency of the renormalization procedure from a dynamical point of view.

c) Fixed point formulation. We are now able to formulate equation (V.8) as a fixed point problem. We define on $\mathbf{C}^{\beta}_{\alpha,T}(X)$ a map Φ setting

$$\Phi\Big((u, u')\Big) = (v, v')$$

where

$$v' := \frac{B(g(u)) - \Big\{A(f(u)) - A(f(u_0^T))\Big\}u'}{A(f(u_0^T))}$$

and v solves the equation

$$\begin{aligned}\partial_t v - A(f(u_0^T))\Delta v = \mathsf{P}_{A(f(u_0^T))v'}\xi + \Big\{A(f(u)) - A(f(u_0^T))\Big\}\Delta u^\sharp + g'(u)\,\Pi\Big(B(u^\sharp), \xi\Big) \\ - u' f'(u)\,\Pi\Big(A(u^\sharp), \xi\Big) + \sharp_1(u, u')\end{aligned} \tag{V.12}$$

with the initial value

$$v(0) = u_0.$$

Note that in view of Lemma V.4 below, (v, v') is an element of $\mathbf{C}^{\beta}_{\alpha,T}(X)$. Furthermore the equation on the remainder $v^\sharp$ writes

$$\begin{aligned}\partial_t v^\sharp - A(f(u_0^T))\Delta v^\sharp = \Big\{A(f(u)) - A(f(u_0^T))\Big\}\Delta u^\sharp + g'(u)\,\Pi\Big(B(u^\sharp), \xi\Big) \\ - u' f'(u)\,\Pi\Big(A(u^\sharp), \xi\Big) + \sharp_2(u, u')\end{aligned} \tag{V.13}$$

with initial value

$$v^\sharp(0) = u_0 - \mathsf{P}_{v'(0)}X = u_0 - \overline{\mathsf{P}}_{v'(0)}X.$$

Denote by $\mathcal{B}_T(r)$ the following ball of $\mathbf{C}^{\beta}_{\alpha,T}(X)$

$$\mathcal{B}_T(r) := \left\{(u, u') \in \mathbf{C}^{\beta}_{\alpha,T}(X)\,;\, u(0) = u_0\,,\, u'(0) = \frac{B(g(u_0))}{A(f(u_0))}\,,\, \|u\|_{\mathbf{C}^{\beta}_{\alpha,T}(X)} \le r\right\}.$$

We prove below that for a sufficiently large radius r and sufficiently small time horizon T the map Φ is a contraction of $\mathcal{B}_T(r)$.

V.2.2 Short time estimate.

The technical part of the argument is handled by the following Schauder-like estimate found in Lemma 5 of [BDH16].

Lemma V.3 – *Pick $u_0 \in \mathcal{C}^\alpha(\mathbb{T}^2)$ and $w \in C^2(\mathbb{T}^2)$ bounded below by some positive* *constant. Let also ϕ_1, ϕ_2 be such that*

$$\phi_1 \in C\Big((0,T], \mathcal{C}^{2\beta-2}(\mathbb{T}^2)\Big) \quad \textit{and} \quad \ell_1(T) := \sup_{0<t\leq T} t^{\frac{2\beta-\alpha}{2}} \|\phi_1(t)\|_{\mathcal{C}^{2\beta-2}} < +\infty,$$

and

$$\phi_2 \in C\Big((0,T], \mathcal{C}^{\alpha+\beta-2}(\mathbb{T}^2)\Big) \quad \textit{and} \quad \ell_2(T) := \sup_{0<t\leq T} t^{\frac{2\beta-\alpha}{2}} \|\phi_2(t)\|_{\mathcal{C}^{\alpha+\beta-2}} < +\infty.$$

Then for T small enough the solution of the equation

$$\partial_t u - w\Delta u = \phi_1 + \phi_2$$

with initial data u_0 satisfies the estimate

$$\|u\|_{\mathscr{C}^\alpha} + \sup_{0<t\leq T} t^{\frac{2\beta-\alpha}{2}} \|u(t)\|_{\mathcal{C}^{2\beta}} \lesssim \|u_0\|_{\mathcal{C}^\alpha} + \ell_1(T) + T^{\frac{\alpha-\beta}{2}} \ell_2(T),$$

where the implicit constant depends only on $\|w\|_{\mathcal{C}^\alpha}$.

We obtain as a consequence of Lemma V.3 the stability of the mapping Φ as a function from the space of paracontrolled functions $\mathbf{C}^\beta_{\alpha,T}(X)$ into itself.

Lemma V.4 – *Let u_0, w, ϕ_1, ϕ_2 be as in Lemma V.3 and let $u' \in \mathcal{C}^\beta$. Denote by u the solution to*

$$\partial_t u - w\Delta u = \mathsf{P}_{wu'}\xi + \phi_1 + \phi_2$$

starting from $u(0) = u_0$. Then $(u, u') \in \mathbf{C}^\beta_{\alpha,T}(X)$.

Proof – Set $u^{\#} := u - \overline{\mathsf{P}}_{u'}X$, it only remains to check that $u^\sharp$ has the regularity matching the definition (V.9) to ensure that (u, u') has indeed a paracontrolled structure. Note that in view of Lemma A.13 the term

$$(\partial_t - w\Delta)\overline{\mathsf{P}}_{u'}X - \mathsf{P}_{wv'}(-\Delta X)$$

belongs to $C_T\mathcal{C}^{\alpha+\beta-2}$. As such $u^\sharp$ satisfies the equation

$$\begin{aligned}
\partial_t u^\sharp - w\Delta u^\sharp &= \partial_t u - w\Delta u - (\partial_t - w\Delta)\Big(\overline{\mathsf{P}}_{u'}X\Big) \\
&= \mathsf{P}_{wu'}\xi + \phi_1 + \phi_2 - \Big((\partial_t - w\Delta)\overline{\mathsf{P}}_{u'}X - \mathsf{P}_{wv'}(\xi)\Big) - \mathsf{P}_{wv'}\xi \\
&=_{\alpha+\beta-2} \phi_1 + \phi_2
\end{aligned}$$

starting from $u^\sharp(0)$. Lemma V.3 then finishes the proof. ▷

First to control v', let $(u_1, u_1'), (u_2, u_2') \in \mathbf{B}(T, r)$, we control the difference $v_1' - v_2'$ writing

$$A(f(u_0^T))\Big(v_1' - v_2'\Big) = B(g(u_1)) - B(g(u_2)) - \Big\{A(f(u_1)) - A(f(u_0^T))\Big\}u_1' + \Big\{A(f(u_2)) - A(f(u_0^T))\Big\}u_2'.$$

First use the fact that B gains $-s$ derivatives and Lemma I.4 to get

$$\begin{aligned}\Big\|B(g(u_1)) - B(g(u_2))\Big\|_{\mathscr{C}^\beta} &\lesssim \Big\|B(g(u_1)) - B(g(u_2))\Big\|_{\mathscr{C}^{\beta-s}} \\ &\lesssim \Big(1 + \|u_1\|_{\mathscr{C}^\beta}\Big)\|u_1 - u_2\|_{\mathscr{C}^\beta} \\ &\lesssim T^{\frac{\alpha-\beta}{2}}\Big(1 + \|u_1\|_{\mathscr{C}^\alpha}\Big)\|u_1 - u_2\|_{\mathscr{C}^\alpha},\end{aligned}$$

where we used the assumption that $\beta - s < \alpha$. Then, writing

$$A_i := A(f(u_i)) - A(f(u_0^T)),$$

we have

$$\begin{aligned}\Big\|A_1 u_1' - A_2 u_2'\Big\|_{\mathscr{C}^\beta} &\lesssim \|A_1(u_1' - u_2')\|_{\mathscr{C}^\beta} + \|(A_1 - A_2)u_2'\|_{\mathscr{C}^\beta} \\ &\lesssim \|A_1\|_{C_TL^\infty}\|u_1' - u_2'\|_{\mathscr{C}^\beta} + \|A_1\|_{\mathscr{C}^\beta}\|u_1' - u_2'\|_{C_TL^\infty} \\ &\quad + \|A_1 - A_2\|_{C_TL^\infty}\|u_2'\|_{\mathscr{C}^\beta} + \|A_1 - A_2\|_{\mathscr{C}^\beta}\|u_2'\|_{C_TL^\infty}.\end{aligned}$$

The difference $A_1 - A_2 = A(f(u_1)) - A(f(u_2))$ is dealt with in the same way as B above, with A_i estimated per se as we cannot extract a contracting factor from the difference $\|u_1' - u_2'\|_{\mathscr{C}^\beta}$. We have

$$A_i = \Big\{A(f(u_i)) - A\Big(f(u(0))\Big)\Big\} + \Big\{A\Big(f(u(0))\Big) - A(f(u_0^T))\Big\},$$

thus resonning as for the B term, we use the continuity properties of A and the paralinearization Lemma I.4 to obtain

$$\Big\|A(f(u_i)) - A\Big(f(u(0))\Big)\Big\|_{\mathscr{C}^\beta} \lesssim T^{\frac{\alpha-\beta}{2}}\Big(1 + \|u_i\|_{\mathscr{C}^\alpha}\Big)\|u_i - u(0)\|_{\mathscr{C}^\alpha}$$

and in a similar way, using the classical estimate (A.13) on $\mathrm{Id} - e^{T\Delta}$ one gets

$$\Big\|A\Big(f(u(0))\Big) - A(f(u_0^T))\Big\|_{\mathcal{C}^\beta} \lesssim T^{\frac{\alpha-\beta}{2}}\Big(1 + \|u_0\|_{\mathcal{C}^\alpha}\Big)\|u_0\|_{\mathcal{C}^\alpha}.$$

In the end one has

$$\Big\|A_1 u_1' - A_2 u_2'\Big\|_{\mathscr{C}^\beta} \leq T^{\frac{\alpha-\beta}{2}} P\Big(\|u_i\|_{\mathscr{C}^\alpha}, \|u_i'\|_{\mathscr{C}^\beta}, \|u_0\|_{\mathcal{C}^\alpha}\Big)\Big(\|u_1 - u_2\|_{\mathscr{C}^\alpha} + \|u_1' - u_2'\|_{\mathscr{C}^\beta}\Big),$$

for some constant $P(\|u_i\|_{\mathscr{C}^\alpha}, \|u_i'\|_{\mathscr{C}^\beta}, \|u_0\|_\alpha)$ that growths at most polynomially as a function of its arguments.

Now to deal with the remainder term $v^\sharp$. Recall from (V.12) and (V.13) the notations for the functions $\sharp_1$ and $\sharp_2$. Note that $v_1^\sharp - v_2^\sharp$ is a solution of the equation

$$\begin{aligned}\Big(\partial_t - A(f(u_0^T))\Delta\Big)\Big(v_1^\sharp - v_2^\sharp\Big) = &\Big\{A(f(u_1)) - A(f(u_0^T))\Big\}\Delta u_1^\sharp - \Big\{A(f(u_2)) - A(f(u_0^T))\Big\}\Delta u_2^\sharp \\ &+ g'(u_1)\,\mathsf{\Pi}\Big(B(u_1^\sharp),\xi\Big) - g'(u_2)\,\mathsf{\Pi}\Big(B(u_2^\sharp),\xi\Big) \\ &+ u_2' f'(u_2)\,\mathsf{\Pi}\Big(A(u_2^\sharp),\xi\Big) - u_1' f'(u_1)\,\mathsf{\Pi}\Big(A(u_1^\sharp),\xi\Big) \\ &+ \sharp_2(u_1,u_1') - \sharp_2(u_2,u_2')\end{aligned}$$

with null initial condition. Set

$$\phi_1 := \Big\{A(f(u_1)) - A(f(u_0^T))\Big\}\Delta u_1^\sharp - \Big\{A(f(u_2)) - A(f(u_0^T))\Big\}\Delta u_2^\sharp$$

and

$$\begin{aligned}\phi_2 := &\, g'(u_1)\,\mathsf{\Pi}\Big(B(u_1^\sharp),\xi\Big) - g'(u_2)\,\mathsf{\Pi}\Big(B(u_2^\sharp),\xi\Big) \\ &+ u_2' f'(u_2)\,\mathsf{\Pi}\Big(A(u_2^\sharp),\xi\Big) - u_1' f'(u_1)\,\mathsf{\Pi}\Big(A(u_1^\sharp),\xi\Big) + \sharp_2(u_1,u_1') - \sharp_2(u_2,u_2')\end{aligned}$$

We now check that ϕ_1 and ϕ_2 satisfy the regularity assumptions of Lemma V.3. Concerning $\phi_1(t)$, the estimate in $\mathcal{C}^{2\beta-2}(\mathbb{T}^2)$ is similar to what we did for $v_1' - v_2'$ so one has

$$\|\phi_1(t)\|_{\mathcal{C}^{2\beta-2}} \lesssim T^{\frac{\alpha-\beta}{2}} Q(\|u_i\|_{\mathscr{C}^\alpha}, \|u_0\|_{\mathcal{C}^\alpha})\left(\|u_1-u_2\|_{\mathscr{C}^\alpha}\|u_1^\sharp(t)\|_{\mathcal{C}^{2\beta-2}} + \|u_1^\sharp(t) - u_2^\sharp(t)\|_{\mathcal{C}^{2\beta-2}}\right)$$

and therefore

$$\sup_{0<t\leq T} t^{\frac{2\beta-\alpha}{2}}\|\phi_1(t)\|_{\mathcal{C}^{2\beta-2}} \lesssim T^{\frac{\alpha-\beta}{2}} Q\Big(\|u_i\|_{\mathscr{C}^\alpha}, \|u_0\|_{\mathcal{C}^\alpha}\Big)\|u_1-u_2\|_{\mathbf{C}^\beta_{\alpha,T}(X)}$$

for yet another positive constant $Q(\cdots)$ with polynomial growth with respect to its parameters.

For ϕ_2, as the function $\sharp_2$ is locally lipschitz in (u,u') it will not cause any issue in the estimate and only the first two terms need a special treatment as they involve a $u^\sharp$ factor. We only consider the B-term as they are both dealt with in the same way. We need to estimate

$$\Big\|\Big(g'(u_1)\,\mathsf{\Pi}\Big(B(u_1^\sharp),\xi\Big) - g'(u_2)\,\mathsf{\Pi}\Big(B(u_2^\sharp),\xi\Big)\Big)(t)\Big\|_{\mathcal{C}^{\alpha+\beta-2}}$$

for a positive time t. An elementary splitting gives for this quantity the upper bound

$$\begin{aligned}\Big\|\Big(g'(u_1) - g'(u_2)\Big)(t)\,\mathsf{\Pi}\Big(B(u_1^\sharp),\xi\Big)(t)\Big\|_{\mathcal{C}^{\alpha+\beta-2}} + \Big\|\Big(g'(u_1) - g'(u_2)\Big)(t)\Big\|_{\mathcal{C}^\beta}\|u_1^\sharp(t)\|_{\mathcal{C}^{2\beta}} \\ + \|g'(u_2)(t)\|_{\mathcal{C}^\beta}\Big\|(u_1^\sharp - u_2^\sharp)(t)\Big\|_{\mathcal{C}^{2\beta}}.\end{aligned}$$

Note that as g is C_b^3 one has

$$\|g'(u_2)(t)\|_{\mathcal{C}^\beta} \leq C\Big(\|u_2\|_{\mathbf{C}^\beta_{\alpha,T}(X)}\Big)$$

and

$$\Big\|\Big(g'(u_1) - g'(u_2)\Big)(t)\Big\|_{\mathcal{C}^\beta} \leq C\Big(\|u_i\|_{\mathbf{C}^\beta_{\alpha,T}(X)}\Big)\|u_1 - u_2\|_{\mathbf{C}^\beta_{\alpha,T}(X)}$$

for yet again polynomial constants. One therefore has

$$\sup_{0<t\leq T} t^{\frac{2\beta-\alpha}{2}} \|\phi_2(t)\|_{\mathcal{C}^{\alpha+\beta-2}} \leq C\Big(\|u_i\|_{\mathbf{C}^\beta_{\alpha,T}(X)}\Big)\|u_1 - u_2\|_{\mathbf{C}^\beta_{\alpha,T}(X)}.$$

Finally one can apply the modified Schauder estimate from Lemma V.3 together with the classical Schauder estimates to recover an estimate on $v_1 - v_2$

$$\begin{aligned} \|v_1 - v_2\|_{\mathbf{C}^\beta_{\alpha,T}(X)} &= \|v_1' - v_2'\|_{\mathscr{C}^\beta} + \Big\|v_1^\sharp - v_2^\sharp\Big\|_{\mathscr{C}^\alpha} + \sup_{0<t\leq T} t^{\frac{2\beta-\alpha}{2}} \Big\|u_1^\sharp(t) - u_2^\sharp(t)\Big\|_{\mathcal{C}^{2\beta}} \\ &\leq C\Big(\|u_1\|_{\mathbf{C}^\beta_{\alpha,T}(X)}, \|u_2\|_{\mathbf{C}^\beta_{\alpha,T}(X)}\Big)\, T^{\frac{\alpha-\beta}{2}} \|u_1 - u_2\|_{\mathbf{C}^\beta_{\alpha,T}(X)} \end{aligned} \tag{V.14}$$

with the constant $C(\cdot)$ depending continuously on the enhanced noise and polynomially on $u_i \in \mathbf{C}^\beta_{\alpha,T}(X)$. Here it is crucial that in Lemma V.3 the constant depends only on the $\mathcal{C}^\alpha$ norm of w as we want to an estimate on $w = u_0^T$ that does not explode as $T > 0$ goes to 0. To conclude note that

$$u := \overline{\mathsf{P}}_{\frac{B(g(u_0))}{A(f(u_0))}} X + u_0^T$$

defines an element of $\mathbf{C}^\beta_{\alpha,T}(X)$ so if $\widetilde{u} \in \mathbf{B}(T, r)$ and

$$\max\Big(\|u\|_{\mathbf{C}^\beta_{\alpha,T}(X)}, \|\Phi(u)\|_{\mathbf{C}^\beta_{\alpha,T}(X)}\Big) < \frac{r}{2}$$

then

$$\|\Phi(\widetilde{u})\|_{\mathbf{C}^\beta_{\alpha,T}(X)} \leq r/2 + 2rC(r)T^{\frac{\alpha-\beta}{2}} \leq r$$

if T is chosen small enough, depending on r. This means that for r large enough and T small enough, both depending only on the size of the initial value u_0 and the problem data

$$\Big(\xi, \mathsf{\Pi}\Big(A(X), \xi\Big), \mathsf{\Pi}\Big(B(X), \xi\Big), f, g\Big),$$

the map Φ sends the ball $\mathbf{B}(T, r)$ into itself and is a contraction from (V.14). As such it has a unique fixed point in that ball, and the equation

$$\partial_t u - A(f(u))\Delta u = B(g(u))\, \xi$$

with initial condition $u(0) = u_0 \in \mathcal{C}^\alpha(\mathbb{T}^2)$ has a unique paracontrolled solution on a small time interval $[0, T]$.

V.3 – Renormalization matters

We construct in Subsection V.3.1 the enhanced noise; it was used as a data in the analytic setting of Section V.2. The renormalization process used for that purpose comes with a dynamical interpretation of the solution to the fixed point problem for Φ. The renormalized equation is described in Section V.3.2.

V.3.1 Enhanced noise

Following the expansions and given an arbitrary symbol a of order $s \leq 0$ we need to make sense of the resonant product $\mathsf{\Pi}\big(A(X), \xi\big)$ as an element of $\mathcal{C}^{2\alpha-s-2}(\mathbb{T}^2)$. This is done via a renormalization procedure described here. Let $\chi : [0, \infty) \to [0, 1]$ be of class C^∞, equal to 1 on $[0, 1]$ and 0 on $[2, \infty)$. Set for any distribution u on $\mathbb{T}^2$

$$R^\varepsilon(u) := \mathcal{F}^{-1}\Big(\chi(\varepsilon|\cdot|)\, \mathcal{F}(u)(\cdot)\Big)$$

and recall that we regularize the space white noise ξ into a smooth function using that smooth Fourier cut-off function

$$\xi^\varepsilon := R^\varepsilon(\xi).$$

Theorem V.5 – *For any $a \in \Sigma_0^s$ with associated operator A there exists some deterministic diverging function $c : \varepsilon \in (0, 1] \to C^\infty(\mathbb{T}^2)$ such that*

$$\mathsf{\Pi}\Big(A(X^\varepsilon), \xi^\varepsilon\Big) - c^\varepsilon$$

converges in $L^r\Big(\Omega; \mathcal{C}^{2\alpha-2-s}(\mathbb{T}^2)\Big)$ for any $1 \leq r < \infty$ to a limit $:\!\mathsf{\Pi}(AX, \xi)\!:$ in $\mathcal{C}^{2\alpha-2-s}(\mathbb{T}^2)$.

Proof – First note that

$$\xi^\varepsilon \xrightarrow{\mathcal{C}^{\alpha-2}} \xi \quad \text{and} \quad X^\varepsilon \xrightarrow{\mathcal{C}^{\alpha}} X$$

almost surely and in $L^r(\Omega)$ for any $1 \leq r < \infty$. This is due to continuity of $(-\Delta)^{-1}$ for X^ε, and the hypercontractivity and Besov embedding arguments used below can also be used to prove the convergence of ξ^ε. Note that due to the commutator estimate from Proposition 3.7 in [BDFT23b], $[A, R^\varepsilon](X) \in \mathcal{C}^{2\alpha-s}(\mathbb{T}^2)$. This brings us back to looking at

$$\mathsf{\Pi}\Big((A(X))^\varepsilon, \xi^\varepsilon\Big) - c^\varepsilon$$

which is what we look at below. Note that if $\widehat{a} = \widehat{a}^x$ is the Fourier transform of a with respect to x, then

$$(AX)^\varepsilon = \sum_{n\in\mathbb{Z}^2} \chi(\varepsilon|n|)\widehat{AX}(n)e_n = \sum_{n,p\in\mathbb{Z}^2} \chi(\varepsilon|n|)\widehat{X}(n-p)\widehat{a}(p, n-p)e_n =: \sum_{p\in\mathbb{Z}^2} (AX)_p^\varepsilon,$$

where e_n is the function $e_n(x) = e^{in\cdot x}$. We shall carry out the renormalization pro-

cedure for each $(AX)^\varepsilon_p$ and then pass to the limit by keeping careful track of the p-dependency.
Fix some $p \in \mathbb{N}$ and $\varepsilon > 0$, the very definition of Π yields

$$\Pi\Big((AX)^\varepsilon_p, \xi^\varepsilon\Big) = \sum_{|i-j|\le 1} \sum_{k,l\in\mathbb{Z}^2, l\neq p} \rho_i(k)\, \rho_j(l)\, \chi(\varepsilon|k|)\, \chi(\varepsilon|l|)\, \frac{\widehat{a}(p, l-p)}{|l-p|^2}\, \widehat{\xi}(k)\, \widehat{\xi}(l-p) e_{k+l}.$$

Set

$$c_p(\varepsilon) := \mathbb{E}\left[\Pi\Big((AX)^\varepsilon_p, \xi^\varepsilon\Big)\right] = \sum_{|i-j|\le 1} \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \rho_i(k)\, \rho_j(p-k)\, \chi(\varepsilon|k|)\, \chi(\varepsilon|p-k|)\, \frac{\widehat{a}(p,k)}{|k|^2}\, e_p \tag{V.15}$$

and

$$Y^\varepsilon_p := (AX)^\varepsilon_p - c_p(\varepsilon).$$

We prove that Y^ε_p converges towards some Y_p in $\mathcal{C}^{2\alpha-2}(\mathbb{T}^2)$, in $L^r(\Omega)$ for any $r \ge 1$.
Let $q \ge -1$ and $\varepsilon, \eta > 0$. Then

$$\begin{aligned}\Delta_q(Y^\varepsilon_p - Y^\eta_p) \quad &= \quad \sum_{\substack{|i-j|\le 1\\ k,l\in\mathbb{Z}^2, l\neq p}} \rho_q(k+l)\, \rho_i(k)\, \rho_j(l)\, \frac{\widehat{a}(p, l-p)}{|l-p|^2}\Big(\chi(\varepsilon|k|)\, \chi(\varepsilon|l|) - \chi(\eta|k|)\, \chi(\eta|l|)\Big) \\ &\qquad \times \Big(\widehat{\xi}(k)\, \widehat{\xi}(l) - \delta_{k,p-l}\Big)\, e_{k+l}.\end{aligned}$$

When computing the expectation $\mathbb{E}\Big[|\Delta_q(Y^\varepsilon_p - Y^\eta_p)|^2\Big]$ we will come across a term in the form of

$$\mathbb{E}\left[\Big(\widehat{\xi}(k)\, \widehat{\xi}(l-p) - \delta_{k,p-l}\Big)\Big(\overline{\widehat{\xi}(k')}\, \overline{\widehat{\xi}(l'-p)} - \delta_{k',p-l'}\Big)\right].$$

This can be handled using the properties of the kernel of $\widehat{\xi}(\cdot)$ and Isserlis' formula as $\widehat{\xi}(k)$ are gaussian

$$\mathbb{E}\left[\Big(\widehat{\xi}(k)\, \widehat{\xi}(l-p) - \delta_{k,p-l}\Big)\Big(\overline{\widehat{\xi}(k')}\, \overline{\widehat{\xi}(l'-p)} - \delta_{k',p-l'}\Big)\right] = \delta_{k,k'}\, \delta_{l,l'} + \delta_{k,l'-p}\, \delta_{l,k'+p}$$

hence we can keep only the terms where $(k,l) \in \{(k',l'), (l'-p, k'+p)\}$. This allows to get rid of the $e_{k+l-k'-l'}$ terms and get

$$\begin{aligned}\mathbb{E}\left[\Big|\Delta_q(Y^\varepsilon_p - Y^\eta_p)\Big|^2\right] &= \sum_{\substack{|i_1-j_1|\le 1\\ |i_2-j_2|\le 1}} \sum_{\substack{k,l\in\mathbb{Z}^2, l\neq p\\ k',l'\in\mathbb{Z}^2, l'\neq p}} \rho_q(k+l)^2\, \rho_{i_1}(k)\, \rho_{i_2}(k')\, \rho_{j_1}(l)\, \rho_{j_2}(l') \\ &\times \Big\{\chi(\varepsilon|k|)\, \chi(\varepsilon|l|) - \chi(\eta|k|)\, \chi(\eta|l|)\Big\}\Big\{\chi(\varepsilon|k'|)\chi(\varepsilon|l'|) - \chi(\eta|k'|)\chi(\eta|l'|)\Big\} \\ &\times \frac{1}{|l-p|^2}\, \frac{1}{|l'-p|^2}\, \widehat{a}(p, l-p)\, \overline{\widehat{a}(p, l'-p)}\Big(\delta_{k,k'}\, \delta_{l,l'} + \delta_{k,l'-p}\, \delta_{l,k'+p}\Big).\end{aligned}$$

Since in both sums $|i-j| \leq 1$ and $\mathrm{supp}(\rho)$ is a fixed annulus, we can actually keep only the terms where $|k| \sim |l|$ (resp. $|k'| \sim |l'|$), that is $\theta|l| \leq |k| \leq \theta^{-1}|l|$ for some $\theta > 0$ that does not depend on i_1, i_2, j_1, j_2 and bound the sum on i_1, i_2, j_1, j_2 by a constant independent on k, l, k', l', leaving us with an upper bound for $\mathbb{E}\left[\left|\Delta_q(Y_p^\varepsilon - Y_p^\eta)(x)\right|^2\right]$ of the form

$$\sum_{\substack{k,l\in\mathbb{Z}^2, l\neq p\\ \theta|l|\leq|k|\leq\theta^{-1}|l|}} \rho_q(k+l)^2 \frac{\Psi_1^{\varepsilon,\eta}(k,l)}{|l-p|^4} \left|\widehat{a}(p,l-p)\right|^2 + \sum_{\substack{k,l\in\mathbb{Z}^2, k\neq 0, l\neq p\\ \theta|l|\leq|k|\leq\theta^{-1}|l|}} \rho_q(k+l)^2 \frac{\Psi_2^{\varepsilon,\eta}(k,l,p)}{|k|^2|l-p|^2} \left|\widehat{a}(p,l-p)\widehat{a}(p,k)\right|$$

where

$$\Psi_1^{\varepsilon,\eta}(k,l) = \left|\chi(\varepsilon|k|)\,\chi(\varepsilon|l|) - \chi(\eta|k|)\,\chi(\eta|l|)\right|^2$$

and

$$\Psi_2^{\varepsilon,\eta}(k,l,p) = \Big|\chi(\varepsilon|k|)\,\chi(\varepsilon|l|) - \chi(\eta|k|)\,\chi(\eta|l|)\Big| \Big|\chi(\varepsilon|l+p|)\,\chi(\varepsilon|k-p|) - \chi(\eta|l+p|)\,\chi(\eta|k-p|)\Big|.$$

Apply Estimate (V.4) for $k = l - p$, we have

$$\left|\frac{\widehat{a}(p,l-p)}{|l-p|^2}\right| \lesssim \langle p\rangle^{-4} \langle l\rangle^{s-2} \mathbf{1}_{|p|\leq\mu(1+|l-p|)}$$

so that, up to the $\langle p\rangle^{-4}$ factor, p only shows up in $\Psi_2 \mathbf{1}_{|p|\leq\mu(1+|l-p|)}$. Note that thanks to the size constraint on $|p|$ and χ being smooth and bounded, we can easily derive a bound for Ψ_1 and Ψ_2 (uniformly in p) as

$$|\Psi_1^{\varepsilon,\eta}(k,l)| \vee |\Psi_2^{\varepsilon,\eta}(k,l,p)| \lesssim_\gamma |\varepsilon-\eta|^\gamma \Big(|k|^\gamma + |l|^\gamma\Big)$$

for any $\gamma \in [0,2]$. Fix $\delta > 0$ to be a small parameter and write $n = k + l$. As $|k| \leq \theta^{-1}|l|$, $|n| \leq (1+\theta^{-1})|l|$, this yields

$$\begin{aligned}
\mathbb{E}\left[\left|\Delta_q(Y_p^\varepsilon - Y_p^\eta)(x)\right|^2\right] &\lesssim \frac{|\varepsilon-\eta|^\gamma}{\langle p\rangle^8} \sum_{n\in\mathbb{Z}^2} \rho_q(n)^2 \sum_{\substack{k+l=n\\ \theta|l|\leq|k|\leq\theta^{-1}|l|}} \frac{|k|^\gamma + |l|^\gamma}{\langle l\rangle^{4-2s}} \\
&\lesssim \frac{|\varepsilon-\eta|^\gamma}{\langle p\rangle^8} \sum_{n\in\mathbb{Z}^2} \frac{\rho_q(n)^2}{\langle n\rangle^{2-\delta-2s}} \sum_{\substack{k+l=n\\ \theta|l|\leq|k|\leq\theta^{-1}|l|}} \frac{1}{\langle l\rangle^{2+\delta-\gamma}} \\
&\lesssim \frac{|\varepsilon-\eta|^\gamma}{\langle p\rangle^8} \sum_{n\in\mathbb{Z}^2} \frac{\rho_q(n)^2}{\langle n\rangle^{2-\delta-2s}}
\end{aligned}$$

where the last inequality is valid as soon as $\delta > \gamma$, say $\gamma = \delta/2$. Then it is only a matter of counting how many $\mathbb{Z}^2$ points belong to a 2^q-sized annulus to finally obtain

the bound
$$\mathbb{E}\left[\left|\Delta_q(Y_p^\varepsilon - Y_p^\eta)(x)\right|^2\right] \lesssim 2^{q(\delta+2s)}|\varepsilon-\eta|^{\delta/2}\langle p\rangle^{-8}$$
for any $\delta > 0$. Using Gaussian hypercontractivity, this yields
$$\mathbb{E}\left[\left|\Delta_q(Y_p^\varepsilon - Y_p^\eta)(x)\right|^r\right] \lesssim 2^{q\frac{r}{2}(\delta+2s)}|\varepsilon-\eta|^{r\delta/4}\langle p\rangle^{-4r}$$
so that multiplying both sides by $2^{qr(2\alpha-2-s+2/r)}$ and summing over $q \geq -1$, we recover the $\mathcal{B}^{2\alpha-2+2/r}_{r,r}(\mathbb{T}^2)$ norm
$$\mathbb{E}\Big[\big\|Y_p^\varepsilon - Y_p^\eta\big\|^r_{\mathcal{B}^{2\alpha-2-s+2/r}_{r,r}}\Big] \lesssim \langle p\rangle^{-4r}\left(\sum_{q\geq -1} 2^{qr(2\alpha-2+\delta/2+2/r)}\right)|\varepsilon-\eta|^{r\delta/4}$$
where the series converges if $r > \frac{2}{1-\alpha}$ and $\delta < 2-2\alpha$ for instance. Combined with the continuous embedding of $\mathcal{B}^{2\alpha-2-s+2/r}_{r,r}(\mathbb{T}^2)$ into $\mathcal{C}^{2\alpha-2-s}(\mathbb{T}^2)$ this yields the estimate
$$\mathbb{E}\left[\left\|Y_p^\varepsilon - Y_p^\eta\right\|^r_{\mathcal{C}^{2\alpha-2-s}}\right] \lesssim \langle p\rangle^{-4r}\,|\varepsilon-\eta|^{r\frac{\delta}{4}} \tag{V.16}$$
and proves that (Y_p^ε) is a Cauchy, therefore convergent, sequence in $L^r(\Omega;\mathcal{C}^{2\alpha-2-s}(\mathbb{T}^2))$ for any r large enough – hence for any $r \geq 1$ from hypercontractivity. This proves the existence of the renormalized product $:\mathsf{\Pi}((AX)_p,\xi):$ for each $p \in \mathbb{Z}^2$. Note that we kept explicit track of p. The dominated convergence theorem ensures that the renormalization tranfers to the whole series $\sum_p Y_p^\varepsilon$. This gives the $L^r(\Omega)$ convergence in $\mathcal{C}^{2\alpha-2-s}(\mathbb{T}^2)$ of the renormalized product for all $1 \leq r < \infty$. $\rhd$

Remark 10 – *Note that applying the Kolmogorov continuity criterion on estimate (V.16) for $r\delta > 2$ yields that there exists a modification of the process $\varepsilon \mapsto Y_p^\varepsilon$ that is a continuous function. The convergence of the renormalized product then also happens almost surely, if we allow for a change of the probability space.*

V.3.2 Renormalized equation

As for the convergence part of the claim, fix $u_0 \in \mathcal{C}^\alpha$, note that the fixed point argument we run holds locally uniformly in the enhanced noise data
$$\Xi := \Big(\xi, \mathsf{\Pi}\big(A(X),\xi\big), \mathsf{\Pi}\big(B(X),\xi\big)\Big),$$
hence so does the fixed point solution we obtain. We shall denote by
$$\Xi \mapsto \mathscr{S}(\Xi)$$
the solution mapping corresponding to the paracontrolled equation starting from u_0; it is a continuous function of the enhanced noise data Ξ. If ζ is a smooth noise, ζ has a natural enhancement Z. Write $\widetilde{\mathscr{S}} : \zeta \mapsto v$ for the solution map corresponding to the

locally in time well-posed quasilinear equation

$$\partial_t v - A(f(v))\Delta u = B(g(v))\zeta$$

with initial condition u_0. The map $\mathscr{S}$ extends $\widetilde{\mathscr{S}}$ in so far as

$$\mathscr{S}(Z) = \widetilde{\mathscr{S}}(\zeta).$$

Denote by

$$\zeta \in C^\infty([0,T]\times\mathbb{T}^2) \mapsto \mathscr{S}^\varepsilon(\zeta)$$

the solution map corresponding to the locally in time well-posed equation

$$\partial_t u^\varepsilon - A\Big(f(u)\Big)\Delta u = B\Big(g(u)\Big)\,\zeta + c_a(\varepsilon)\left(\frac{B\Big(g(u)\Big)}{A\Big(f(u)\Big)}\right)^2 f'(u) - c_b(\varepsilon)\frac{B\Big(g(u)\Big)}{A\Big(f(u)\Big)}\,g'(u)$$

with initial condition u_0. Then we have the identity

$$\mathscr{S}^\varepsilon(\xi^\varepsilon) = \mathscr{S}\Big(\xi^\varepsilon, \mathsf{\Pi}\Big(A(X^\varepsilon),\xi^\varepsilon\Big) - c_a(\varepsilon), \mathsf{\Pi}\Big(B(X^\varepsilon),\xi^\varepsilon\Big) - c_b(\varepsilon)\Big).$$

Indeed, in view of Equations (V.10) and (V.11), replacing the formal resonant products $\mathsf{\Pi}(B(X),\xi)$ and $\mathsf{\Pi}(A(X),\xi)$ therein by their approximation respectively $\mathsf{\Pi}\Big(B(X^\varepsilon),\xi^\varepsilon\Big) - c_b(\varepsilon)$ and $\mathsf{\Pi}\Big(A(X^\varepsilon),\xi^\varepsilon\Big) - c_a(\varepsilon)$ and then regrouping the (well-defined) products together we obtain that Equation (V.8) driven by the noise data

$$\Big(\xi^\varepsilon, \mathsf{\Pi}\Big(A(X^\varepsilon),\xi^\varepsilon\Big) - c_a(\varepsilon), \mathsf{\Pi}\Big(B(X^\varepsilon),\xi^\varepsilon\Big) - c_b(\varepsilon)\Big)$$

writes

$$\partial_t u - A\Big(f(u)\Big)\Delta u = B\Big(g(u)\Big)\,\xi^\varepsilon + c_a(\varepsilon)\left(\frac{B\Big(g(u)\Big)}{A\Big(f(u)\Big)}\right)^2 f'(u) - c_b(\varepsilon)\frac{B\Big(g(u)\Big)}{A\Big(f(u)\Big)}\,g'(u).$$

Note that since $\mathscr{S}$ depends locally uniformly on the noise data, there exists a random time T that does not depend on ε for which the equation above starting from u_0 is well-posed in $\mathscr{C}_T^\alpha$. Together with the continuity of $\mathscr{S}$ and Theorem V.5, this completes the proof of the convergence.

Remark 11 – *Throughout this whole Chapter, we considered symbols a,b with derivative index $s \leq 0$, the analysis being straightforward in that case. A similar analysis should be possible for symbol with $s > 0$ small enough (i.e. operators that involve a loss of regularity).*

If we ask the further condition that the action A and B is diagonal (for instance if A and B are convolution operators), then a strong consequence of Theorem V.5 is that the renormalization function is actually a constant. An interesting question is wether the diverging term we need to substract can be a constant in general.

A Analytical toolbox

Contents

A.1 – Analysis on manifolds and function spaces

Recall that we work on a two-dimensional closed (compact, boundaryless), connected, orientable smooth Riemannian manifold $(\mathcal{M}, \mathrm{g})$, where we fix the metric g once and for all and drop it in the remaining of the paper. We set $\{\varphi_n^\Delta\}_{n\geq 0} \subset C^\infty(\mathcal{M})$ to be a basis of $L^2(\mathcal{M})$ consisting of eigenfunctions of $-\Delta$ associated with the eigenvalue $\lambda_n^\Delta \geq 0$, assumed to be arranged in increasing order: $0 \leq \lambda_0^\Delta < \lambda_1^\Delta \leq \lambda_2^\Delta \leq \ldots$ As for λ_n^Δ, $n \to \infty$, we have the following asymptotic behaviour given by Weyl's law:

$$\frac{\lambda_n^\Delta}{n} \longrightarrow \frac{|\mathcal{M}|}{4\pi} \text{ as } n \to \infty. \tag{A.1}$$

Using the above basis of $L^2(\mathcal{M})$, we can expand any $u \in \mathcal{D}'(\mathcal{M})$ as

$$u = \sum_{n\geq 0} \langle u, \varphi_n^\Delta \rangle_{\mathrm{g}} \varphi_n^\Delta,$$

where $\langle \cdot, \cdot \rangle_{\mathrm{g}}$ denotes the distributional pairing $\mathcal{D}'(\mathcal{M}) \times \mathcal{D}(\mathcal{M}) \to \mathbb{R}$ which coincides with the usual inner product in $L^2(\mathcal{M})$ for distributions which are regular enough. For any $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we thus define the Sobolev spaces

$$W^{s,p}(\mathcal{M}, \mathrm{g}) \stackrel{\text{def}}{=} \Big\{ u \in \mathcal{D}'(\mathcal{M}),\ \|u\|_{W^{s,p}(\mathcal{M})} < \infty \Big\}$$

where

$$\|u\|_{W^{s,p}(\mathcal{M})} \stackrel{\text{def}}{=} \Big\| (1-\Delta)^{\frac{s}{2}} u \Big\|_{L^p(\mathcal{M})} = \Big\| \sum_{n\geq 0} (1+\lambda_n^\Delta)^s \langle u, \varphi_n^\Delta \rangle_{\mathrm{g}} \varphi_n^\Delta \Big\|_{L^p(\mathcal{M})}.$$

When $p = 2$ we write $H^s(\mathcal{M}) \stackrel{\text{def}}{=} W^{s,2}(\mathcal{M})$.

Next, recall that for any self-adjoint elliptic operator $\mathbf{A}$ on $L^2(\mathcal{M})$ with discrete spectrum $\{\lambda_n\}$ and orthonormal basis of eigenfunctions $\{\varphi_n\}$, the functional calculus of

$\mathbf{A}$ is defined for any $\psi \in L^\infty(\mathbb{R})$ by

$$\psi\Big(\mathbf{A}\Big)u = \sum_{n\geq 0} \psi(\lambda_n^{\mathbf{A}})\langle u, \varphi_n^{\mathbf{A}}\rangle \varphi_n^{\mathbf{A}}, \tag{A.2}$$

for all $u \in C^\infty(\mathcal{M})$. This in particular allows us to define the more general class of Besov spaces on $\mathcal{M}$ associated with Δ. First, using the functional calculus, we can define the Littlewood-Paley projectors $\mathbf{Q}_M$ for a dyadic integer $M \in 2^{\mathbb{Z}_{\geq -1}} \stackrel{\text{def}}{=} \{0, 1, 2, 4, ...\}$ as

$$\mathbf{Q}_M = \begin{cases} \psi_0\Big(-(2M)^{-2}\Delta\Big) - \psi_0\Big(-M^{-2}\Delta\Big), \ M \geq 1, \\ \psi_0\Big(-\Delta\Big), \ M = 0, \end{cases} \tag{A.3}$$

where $\psi_0 \in C_0^\infty(\mathbb{R})$ is non-negative and such that $\operatorname{supp}\psi_0 \subset [-1, 1]$ and $\psi_0 \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. With the inhomogeneous dyadic partition of unity $\{\mathbf{Q}_M\}_{M\in 2^{\mathbb{Z}_{-1}}}$, we can then define the Besov norms for $p, q \in [1, \infty]$ and $s \in \mathbb{R}$,

$$\|f\|_{\mathcal{B}^s_{p,q}(\mathcal{M})} \stackrel{\text{def}}{=} \Big\|\langle M\rangle^{qs}\|\mathbf{Q}_M f\|_{L^p(\mathcal{M})}\Big\|_{\ell^q_M} = \Bigg(\sum_{M\in 2^{\mathbb{Z}_{-1}}} \langle M\rangle^{qs}\Big\|\mathbf{Q}_M f\Big\|^q_{L^p(\mathcal{M})}\Bigg)^{\frac{1}{q}} \tag{A.4}$$

for any smooth function f. The last sum is of course understood to be the supremum for $q = \infty$.

Definition A.1 – Fix $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. We define the Besov space $\mathcal{B}^s_{p,q}(\mathcal{M})$ to be the completion of $C^\infty(\mathcal{M})$ for the norm $\|\cdot\|_{\mathcal{B}^s_{p,q}(\mathcal{M})}$. In particular, this defines the Hölder spaces $\mathcal{C}^s(\mathcal{M}) = \mathcal{B}^s_{\infty,\infty}(\mathcal{M})$ for any $s \in \mathbb{R}$.

These function spaces are the natural generalization of the usual Besov spaces on $\mathbb{R}^d$ to the context of closed manifolds. Let us recall the following characterization of these spaces from [ORT23, Proposition 2.5]

Lemma A.2 – *Let (U, V, κ) be a coordinate patch and $\chi \in C_0^\infty(V)$. For any $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, there exist $c, C > 0$ such that for any $u \in C^\infty(\mathcal{M})$,*

$$c\|\chi u\|_{B^s_{p,r}(\mathcal{M})} \leq \|\kappa^\star(\chi u)\|_{B^s_{p,r}(\mathbb{R}^2)} \leq C\|u\|_{B^s_{p,r}(\mathcal{M})}.$$

This allows to transfer the usual linear or nonlinear estimates on Besov spaces on $\mathbb{R}^d$ to the case of Besov spaces on $\mathcal{M}$; see for example Lemma A.3 below, which is taken from [ORT23, Corollary 2.7].

Lemma A.3 – *Let $B^s_{p,q}(\mathcal{M})$ be the Besov spaces defined above. Then the following properties hold.*
(i) *For any $s \in \mathbb{R}$ we have $B^s_{2,2}(\mathcal{M}) = H^s(\mathcal{M})$, and more generally for any $2 \leq p < \infty$ and $\varepsilon > 0$ we have*

$$\|u\|_{B^s_{p,\infty}(\mathcal{M})} \lesssim \|u\|_{W^{s,p}(\mathcal{M})} \lesssim \|u\|_{B^s_{p,2}(\mathcal{M})} \lesssim \|u\|_{B^{s+\varepsilon}_{p,\infty}(\mathcal{M})}.$$

(ii) *Let $s \in \mathbb{R}$ and $1 \leq p_1 \leq p_2 \leq \infty$ and $q_1, q_2 \in [1, \infty]$. Set $r = s - 2\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)$, then for* *any* $f \in B^s_{p_1,q}(\mathcal{M})$ *we have*

$$\|f\|_{B^r_{p_2,q}(\mathcal{M})} \lesssim \|f\|_{B^s_{p_1,q}(\mathcal{M})}.$$

(iii) *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha + \beta > 0$*, and* $p_1, p_2, q_1, q_2 \in [1, \infty]$ *with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \qquad \text{and} \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then for any $f \in B^\alpha_{p_1,q_1}(\mathcal{M})$ *and* $g \in B^\beta_{p_2,q_2}(\mathcal{M})$*, we have* $fg \in B^{\alpha\wedge\beta}_{p,q}(\mathcal{M})$*, and it holds:*

— *If* $\alpha \wedge \beta < 0$*, then*

$$\|fg\|_{B^{\alpha\wedge\beta}_{p,q}(\mathcal{M})} \lesssim \|f\|_{B^\alpha_{p_1,q_1}(\mathcal{M})} \|g\|_{B^\beta_{p_2,q_2}(\mathcal{M})}.$$

— *If* $\alpha \wedge \beta > 0$*, then*

$$\|fg\|_{B^{\alpha\wedge\beta}_{p,q}(\mathcal{M})} \lesssim \|f\|_{B^{\alpha\wedge\beta}_{p_1,q_1}(\mathcal{M})} \|g\|_{B^{\alpha\wedge\beta}_{p_2,q_2}(\mathcal{M})}.$$

From [MW17b], we gather the two following most useful estimates that follow from the previous lemma.

Lemma A.4 – *Let* $\alpha_0, \alpha_1 \mathbb{R}$*,* $p_0, p_1, q_0, q_1 \in [1, +\infty]$*,* $\theta \in [0, 1]$ *and set*

$$\alpha = \theta\alpha_1 + (1-\theta)\alpha_2 \qquad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \qquad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Then

$$\|f\|_{B^\alpha_{p,q}} \lesssim \|f\|^\theta_{B^{\alpha_1}_{p_1,q_1}} \|f\|^{1-\theta}_{B^{\alpha_2}_{p_2,q_2}}.$$

Lemma A.5 – *Let* $\alpha > 0$*,* $r \in \mathbb{N}$ *and* $p_1, p_2, q \in [1, +\infty]$*. Set*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

then

$$\|f^{r+1}\|_{B^\alpha_{p,q}} \lesssim \|f^r\|_{L^{p_1}} \|f\|_{B^\alpha_{p_2,q}}.$$

We will also make use of the following fractional Leibniz rule.

Lemma A.6 – *Let* $r, s < 1$ *with* $r + s > 0$*, and* $t = r + s - 1$*. Then there exists* $C > 0$ *such that for any* $u \in H^s(\mathcal{M})$ *and* $v \in H^r(\mathcal{M})$*, we have the fractional Leibniz rule*

$$\|uv\|_{H^t} \leq C\|u\|_{H^s}\|v\|_{H^r}.$$

Proof – The corresponding estimate in $\mathbb{R}^d$ is proved in [Tam01, Corollary 2.1]. Then the one on $\mathcal{M}$ follows from the one in $\mathbb{R}^d$ through the use of a finite partition of unity and Lemma A.2; see the proof of Lemma A.3 (iii) in [ORT23] for details. ▷

A.2 – Schwartz multipliers of the Laplace-Beltrami operator

We will need to estimate the action of semi-classical multipliers on $\mathcal{M}$ with symbol in $\mathcal{S}(\mathbb{R})$. We start by recalling the following universal bound on their kernel from [ORTW20, Lemma 2.5].

Lemma A.7 – *Let $\psi \in \mathcal{S}(\mathbb{R})$, and for any $h \in (0,1]$ define the kernel*

$$\mathcal{K}_h(x,y) \overset{\text{def}}{=} \sum_{n\geq 0} \psi\Big(h^2\lambda_n^\Delta\Big)\varphi_n(x)\varphi_n(y). \tag{A.5}$$

Then for any $L \geq 0$, there exists $C > 0$ such that for any $h \in (0,1]$ and $x, y \in \mathcal{M}$ it holds

$$\Big|\mathcal{K}_h(x,y)\Big| \leq Ch^{-2}\langle h^{-1}\mathbf{d}(x,y)\rangle^{-L}, \tag{A.6}$$

where $\mathbf{d}$ is the geodesic distance on $\mathcal{M}$. In particular, it holds

$$\Big\|\psi(-h^2\Delta)\Big\|_{L^p(\mathcal{M})\to L^q(\mathcal{M})} \lesssim h^{-2(\frac{1}{p}-\frac{1}{q})} \tag{A.7}$$

and

$$\Big\|\psi(-h^2\Delta)\Big\|_{B^{\sigma_1+\sigma_2}_{p,q}(\mathcal{M})\to B^{\sigma_1}_{p,q}(\mathcal{M})} \lesssim h^{-\sigma_2} \tag{A.8}$$

for any $h \in (0,1]$ and $1 \leq p, q \leq \infty$ and $\sigma_1 \in \mathbb{R}$, $\sigma_2 \geq 0$.

Proof – Estimates (A.6) and (A.7) are proved in [ORTW20, Lemma 2.5]. As for (A.8), it is a straightforward adaptation of the proof of (A.11) below given in [ORTW20, Lemma 2.6], since there only the fact that $e^{t\Delta}$ is a Schwartz multiplier was used, but not the precise form of its symbol; see (2.13) in [ORTW20]. ▷

As a corollary of the estimates above, we can then investigate the behaviour of the solutions to the linear heat equation on $\mathcal{M}$. Indeed, we can use the eigenfunctions expansion to represent the solution of the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, \\ u(0) = u_0 \in \mathcal{D}'(\mathcal{M}), \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \mathcal{M},$$

as the distribution

$$u(t) = e^{t\Delta}u_0 = \sum_{n\geq 0} e^{-t\lambda_n^\Delta}\langle u_0, \varphi_n\rangle\varphi_n.$$

The well-known heat kernel is then the kernel of the above propagator, defined as

$$p_t^{\Delta}(x, y) \stackrel{\text{def}}{=} \sum_{n \geq 0} e^{-t\lambda_n^{\Delta}} \varphi_n(x)\varphi_n(y). \tag{A.9}$$

As a corollary to Lemma A.7, we have the following bounds; see [ORTW20, Lemma 2.6] for a proof.

Lemma A.8 – (i) *For any $L \geq 0$, there exists $C > 0$ such that for any $0 < t \leq 1$ and $x, y \in \mathcal{M}$ it holds*

$$|p_t^{\Delta}(x, y)| \leq C t^{-1} \langle t^{-\frac{1}{2}} \mathbf{d}(x, y) \rangle^{-L}. \tag{A.10}$$

(ii) *For any $1 \leq p \leq q \leq \infty$, any $1 \leq r \leq \infty$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \leq \alpha_2$, there exists $C > 0$ such that for any $0 < t \leq 1$ and any $u \in B^{\alpha_1}_{p,r}(\mathcal{M})$, we have* Schauder's estimate

$$\left\| e^{t\Delta} u \right\|_{B^{\alpha_2}_{q,r}(\mathcal{M})} \leq C t^{-\frac{\alpha_2 - \alpha_1}{2} - (\frac{1}{p} - \frac{1}{q})} \left\| u \right\|_{B^{\alpha_1}_{p,r}(\mathcal{M})}. \tag{A.11}$$

We will also use of the following bound on the kernel of fractional antiderivatives on $\mathcal{M}$.

Lemma A.9 – *For $\sigma \in (0; 2)$, let G_σ be the distributional kernel of $(1 - \Delta)^{-\frac{\sigma}{2}}$. Then G_σ is non-negative distribution, smooth away from the diagonal in $\mathcal{M}^2$, and we have the estimate*

$$G_\sigma(x, y) \lesssim \mathbf{d}(x, y)^{\sigma - 2}. \tag{A.12}$$

Proof – Using the eigenfunction expansion, we can write

$$\begin{aligned} G_\sigma(x, y) &= \sum_{n \geq 0} \frac{\varphi_n(x)\varphi_n(y)}{(1 + \lambda_n^{\Delta})^{\sigma}} = \sum_{n \geq 0} \varphi_n(x)\varphi_n(y) \Gamma\Big(\frac{\sigma}{2}\Big)^{-1} \int_0^\infty t^{\frac{\sigma}{2} - 1} e^{-t(1 + \lambda_n^{\Delta})} dt \\ &= \Gamma\Big(\frac{\sigma}{2}\Big)^{-1} \int_0^\infty t^{\frac{\sigma}{2} - 1} e^{-t} p_t^{\Delta}(x, y) dt, \end{aligned}$$

where Γ is the Gamma function, and the equality holds in the sense of distributions on $\mathcal{M} \times \mathcal{M}$. Note that the last integral converges since we assumed $\sigma > 0$. This shows that G_σ is a non-negative distribution, smooth away from the diagonal, and from the estimate (A.10) on the heat kernel, we get for $0 < \mathbf{d}(x, y) \leq 1$:

$$\begin{aligned} G_\sigma(x, y) &\lesssim \int_0^1 t^{\frac{\sigma}{2} - 2} \langle t^{-\frac{1}{2}} \mathbf{d}(x, y) \rangle^{-10} dt + 1 \\ &\lesssim \int_0^{\mathbf{d}(x,y)^2} t^{\frac{\sigma}{2} + 3} \mathbf{d}(x, y)^{-10} + \int_{\mathbf{d}(x,y)^2}^1 t^{\frac{\sigma}{2} - 2} + 1 \\ &\lesssim \mathbf{d}(x, y)^{\sigma - 2}. \end{aligned}$$

This shows (A.12). ▷

A.3 – Estimates from paracontrolled calculus

We gather in this section useful estimates for the treatment of Chapter V. This is somehow a refined version of the introductory properties we stated in Chapter I, taking into account that the functions also depend on time. To that end, we define Gubinelli, Imkeller & Perkowski's modified paraproduct $\overline{\mathsf{P}}$ on parabolic functions as in [GIP15]. Fix some smooth function $\varphi : \mathbb{R} \to \mathbb{R}_+$ with compact support and integral 1. For $f \in C_T C^\infty$ and $i \geq -1$ define for $0 \leq t \leq T$

$$Q_i(f)(t) := \int_{\mathbb{R}} 2^{2i}\varphi\left(2^{2i}(t-s)\right) f\Big((s \wedge T) \vee 0\Big) ds$$

and

$$\overline{\mathsf{P}}_f g := \sum_{i<j-1} \Delta_i\Big(Q_j(f)\Big)(\Delta_j g).$$

The operator $\overline{\mathsf{P}}$ satisfies the following uniform in time continuity estimates

$$\left\|\left(\overline{\mathsf{P}}_f g\right)(t)\right\|_{\mathcal{C}^\alpha} \lesssim \|f\|_{C_T L^\infty} \|g(t)\|_{\mathcal{C}^\alpha}.$$

Denote by $\mathcal{L}$ the heat operator $\partial_t - \Delta$ and write $\|\cdot\|_{C_T\mathcal{C}^\beta}$ for the natural norm on $C\Big([0,T], \mathcal{C}^\beta(\mathbb{T}^2)\Big)$. Similarly we write $\|\cdot\|_{C_T^{\alpha/2}L^\infty}$ for the natural norm on $C^{\alpha/2}\Big([0,T], L^\infty(\mathbb{T}^2)\Big)$, then the modified paraproduct $\overline{\mathsf{P}}$ satisfies the following commutation estimates (see Lemma 5.1 in [GIP15])

Proposition A.10 – *Pick two regularity exponents $\alpha \in (0,1), \beta \in \mathbb{R}$ and a finite positive time horizon T. For $u \in \mathscr{C}_T^\alpha$ and $v \in C\Big([0,T], \mathcal{C}^\beta(\mathbb{T}^2)\Big)$ one has*

(a)
$$\left\|\mathcal{L}\left(\overline{\mathsf{P}}_u v\right) - \overline{\mathsf{P}}_u(\mathcal{L}v)\right\|_{C_T\mathcal{C}^{\alpha+\beta-2}} \lesssim \|u\|_{\mathscr{C}_T^\alpha} \|v\|_{C_T\mathcal{C}^\beta},$$

(b)
$$\left\|\overline{\mathsf{P}}_u v - \mathsf{P}_u v\right\|_{C_T\mathcal{C}^{\alpha+\beta}} \lesssim \|u\|_{C_T^{\alpha/2}L^\infty} \|v\|_{C_T\mathcal{C}^\beta}.$$

The Schauder estimates for the inverse heat operator $\mathcal{L}^{-1}$ take the following form – see e.g. Lemma 2.5 in [PV23] and references therin.

Proposition A.11 – *(a) For $t>0$, $\alpha \in \mathbb{R}$ and $\delta \geq 0$ we have*

$$\left\|e^{t\Delta}u\right\|_{\mathcal{C}^{\alpha+\delta}} \lesssim t^{-\delta/2}\|u\|_{\mathcal{C}^\alpha} \quad \textit{and} \quad \left\|e^{t\Delta}u\right\|_{\mathcal{C}^\delta} \lesssim t^{-\delta/2}\|u\|_{L^\infty}.$$

(b) For $\alpha \in (0,1)$, $u \in \mathcal{C}^\alpha$, then for any $t \geq 0$ one has

$$\left\|e^{t\Delta}u - u\right\|_{L^\infty} \lesssim t^{\alpha/2}\|u\|_{\mathcal{C}^\alpha}. \tag{A.13}$$

(c) Let $u \in C\Big([0,T], \mathcal{C}^\alpha(\mathbb{T}^2)\Big)$ for some $\alpha \in \mathbb{R}$ and $T>0$, then for any $\gamma \in [0,1)$ and

$t \in (0,T]$ we have

$$t^\gamma \left\| \left(\mathcal{L}^{-1}u\right)(t) \right\|_{\mathcal{C}^{\alpha+2}} \lesssim \sup_{s\in[0,t]} s^\gamma \|u(s)\|_{\mathcal{C}^\alpha}.$$

Moreover if $\alpha \in (-2,0)$ then one has

$$\left\| \mathcal{L}^{-1}u \right\|_{C_T^{(\alpha+2)/2}L^\infty} \lesssim \|u\|_{C_T\mathcal{C}^\alpha}.$$

(d) *For $\alpha \in (0,2)$ we have the T-uniform estimate*

$$\left\| \mathcal{L}^{-1}u \right\|_{\mathscr{C}_T^\alpha} \lesssim (1+T)\|u\|_{C_T\mathcal{C}^{\alpha-2}}.$$

(e) *For $\alpha \in (0,2)$, uniformly in $\varepsilon \in (0,1)$ and T, we have*

$$\left\| \mathcal{L}^{-1}(u) \right\|_{\mathscr{C}_T^\alpha} \lesssim T^\varepsilon \|u\|_{C_T\mathcal{C}^{\alpha-2+2\varepsilon}}.$$

(f) *For $\delta \in [0,1)$, $\alpha \in \mathbb{R}$ and $\beta \in [-\alpha, 2-\alpha)$, one has*

$$\sup_{0\le s\le t} s^\delta \left\| \mathcal{L}^{-1}(w)(s) \right\|_{\mathcal{C}^{\alpha+\beta}} \lesssim T^{\frac{\alpha-\beta}{2}} \sup_{0\le s\le t} s^\delta \|w(s)\|_{\mathcal{C}^{2\alpha-2}}.$$

The following elementary statement is useful to get a small factor and produce contractive maps.

Proposition A.12 – *For $0<\beta<\alpha<2$ and $u \in \mathscr{C}_T^\alpha$ that is null at time 0 one has*

$$\|u\|_{\mathscr{C}_T^\beta} \lesssim T^{\frac{\alpha-\beta}{2}} \|u\|_{\mathscr{C}_T^\alpha}.$$

Finally the following commutation lemma is useful whenever we deal with a weighted heat operator, as is the case of a quasilinear parabolic equation. See Lemma 2 in [BDH16] for a proof.

Lemma A.13 – *For $v \in \mathscr{C}_T^\beta$ and $w \in \mathcal{C}^{2\beta}(\mathbb{T}^2)$ one has*

$$\left\| (\partial_t - w\Delta)\left(\overline{\mathsf{P}}_v X\right) - \mathsf{P}_{wv}\Big(-\Delta X\Big) \right\|_{C_T\mathcal{C}^{\alpha+\beta-2}} \lesssim \Big(1+\|w\|_{\mathcal{C}^{2\beta}}\Big)\|v\|_{\mathscr{C}_T^\beta}\|X\|_{\mathcal{C}^\alpha}.$$

BIBLIOGRAPHY

[AC15] R. Allez and K. Chouk, *The continuous Anderson hamiltonian in dimension two*, Nov. 2015. arXiv: `1511.02718 [math]`.

[Ack04] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, *Math. Z.*, vol. 248, *2*, Oct. 2004.

[AK20] S. Albeverio and S. Kusuoka, *The invariant measure and the flow associated to the Φ^4_3-quantum field model*, *Ann. SCUOLA Norm. Super. - Cl. Sci.*, pp. 1359–1427, Dec. 2020. arXiv: `1711.07108 [math-ph]`.

[AK22] S. Albeverio and S. Kusuoka, *Construction of a non-Gaussian and rotation-invariant Φ^4-measure and associated flow on $\mathbb{R}^3$ through stochastic quantization*, Jul. 2022. arXiv: `2102.08040 [math-ph]`.

[And58] P. W. Anderson, *Absence of Diffusion in Certain Random Lattices*, *Phys. Rev.*, vol. 109, *5*, pp. 1492–1505, Mar. 1958.

[AVG20] S. Albeverio, F. C. D. Vecchi, and M. Gubinelli, *Elliptic stochastic quantization*, *Ann. Probab.*, vol. 48, *4*, pp. 1693–1741, Jul. 2020.

[Bab71] I. Babuška, *Error-bounds for finite element method*, *Numer. Math.*, vol. 16, *4*, pp. 322–333, Jan. 1971.

[Bai23] I. Bailleul, *Uniqueness of the Φ^4_3 measures on closed Riemannian* 3-*manifolds*, Jun. 2023. arXiv: `2306.07616 [math-ph]`.

[BB16] I. Bailleul and F. Bernicot, *Heat semigroup and singular PDEs*, *Journal of Functional Analysis*, vol. 270, *9*, pp. 3344–3452, May 2016.

[BB19] I. Bailleul and F. Bernicot, *HIGH ORDER PARACONTROLLED CALCULUS*, *Forum Math. Sigma*, vol. 7, e44, Jan. 2019.

[BBF17] I. Bailleul, F. Bernicot, and D. Frey, *Space-time paraproducts for paracontrolled calculus, 3d-PAM and multiplicative Burgers equations*, Apr. 2017. arXiv: `1506.08773 [math]`.

[BCCH20] Y. Bruned, A. Chandra, I. Chevyrev, and M. Hairer, *Renormalising SPDEs in regularity structures*, *J. Eur. Math. Soc.*, vol. 23, *3*, pp. 869–947, 2020.

[BCD11] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations* (Grundlehren Der Mathematischen Wissenschaften). Berlin, Heidelberg: Springer, 2011, vol. 343.

[BDFT23a] I. Bailleul, N. V. Dang, L. Ferdinand, and T. D. Tô, Φ^4_3 *Measures on compact Riemannian 3-manifolds*, Oct. 2023. arXiv: `2304.10185 [math-ph]`.

[BDFT23b] I. Bailleul, N. V. Dang, L. Ferdinand, and T. D. Tô, *Global harmonic analysis for Φ^4_3 on closed Riemannian manifolds*, Jul. 2023. arXiv: `2306.07757 [math-ph]`.

[BDH16] I. Bailleul, A. Debussche, and M. Hofmanová, *Quasilinear generalized parabolic Anderson model equation*, Nov. 2016. arXiv: `1610.06726 [math]`.

[BDM23] I. Bailleul, N. V. Dang, and A. Mouzard, *Analysis of the Anderson operator*, Nov. 2023. arXiv: `2201.04705 [math-ph]`.

[BDNY22] B. Bringmann, Y. Deng, A. R. Nahmod, and H. Yue, *Invariant Gibbs measures for the three dimensional cubic nonlinear wave equation*, Jun. 2022. arXiv: `2205.03893 [math-ph]`.

[BDZ23] N. Barashkov, F. C. De Vecchi, and I. Zachhuber, *Invariant Gibbs measure for Anderson NLW*, Oct. 2023. arXiv: `2309.01635 [math-ph]`.

[BE24] I. Bailleul and H. Eulry, *Non-local quasilinear singular SPDEs*, May 2024. arXiv: `2405.18057 [math]`.

[BER22] I. Bailleul, H. Eulry, and T. Robert, *Variational methods for some singular stochastic elliptic PDEs*, Oct. 2022. arXiv: `2202.09628 [math]`.

[BG20] N. Barashkov and M. Gubinelli, *A variational method for Φ^4_3*, *Duke Math. J.*, vol. 169, *17*, pp. 3339–3415, Nov. 2020.

[BG23] N. Barashkov and M. Gubinelli, *On the variational method for Euclidean quantum fields in infinite volume*, *Prob. Math. Phys.*, vol. 4, *4*, pp. 761–801, Nov. 2023. arXiv: `2112.05562 [math-ph]`.

[BGT04] N. Burq, P. Gérard, and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, *Am. J. Math.*, vol. 126, *3*, pp. 569–605, 2004.

[BH21a] I. Bailleul and M. Hoshino, *Paracontrolled calculus and regularity structures I*, *J. Math. Soc. Jpn.*, vol. 73, *2*, pp. 553–595, Apr. 2021.

[BH21b] I. Bailleul and M. Hoshino, *Paracontrolled calculus and regularity structures II*, *J. L'École Polytech. — Mathématiques*, vol. 8, pp. 1275–1328, 2021.

[BHK23] I. Bailleul, M. Hoshino, and S. Kusuoka, *Regularity structures for quasilinear singular SPDEs*, Jan. 2023. arXiv: `2209.05025 [math]`.

[BHZ19] Y. Bruned, M. Hairer, and L. Zambotti, *Algebraic renormalisation of regularity structures*, *Invent. math.*, vol. 215, *3*, pp. 1039–1156, Mar. 2019.

[BK05] J. Bourgain and C. E. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, *Invent. math.*, vol. 161, *2*, pp. 389–426, Aug. 2005.

[BL23] N. Barashkov and P. Laarne, *Invariance of ϕ^4 measure under nonlinear wave and Schrödinger equations on the plane*, Apr. 2023. arXiv: `2211.16111 [math-ph]`.

[BM23] I. Bailleul and A. Mouzard, *Paracontrolled calculus for quasilinear singular PDEs*, *Stoch PDE: Anal Comp*, vol. 11, *2*, pp. 599–650, Jun. 2023.

[Bon81] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, *Ann. Sci. École Norm. Sup.*, vol. 14, *2*, pp. 209–246, 1981.

[Bou93] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations*, 1993.

[Bou94] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, *Commun. Math. Phys.*, vol. 166, *1*, pp. 1–26, Jan. 1994.

[Bou96] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, *Commun. Math. Phys.*, vol. 176, *2*, pp. 421–445, Jan. 1996.

[BP09] N. Burq and F. Planchon, *Global Existence for Energy Critical Waves in 3-D Domains: Neumann Boundary Conditions*, *Am. J. Math.*, vol. 131, *6*, pp. 1715–1742, 2009. JSTOR: `40388566`.

[BT08a] N. Burq and N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations I: local theory*, *Invent. math.*, vol. 173, *3*, pp. 449–475, Sep. 2008.

[BT08b] N. Burq and N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations II: a global existence result*, *Invent. math.*, vol. 173, *3*, pp. 477–496, Sep. 2008.

[BTT18] N. Burq, L. Thomann, and N. Tzvetkov, *Remarks on the Gibbs measures for nonlinear dispersive equations*, *Ann. Fac. Sci. Toulouse Mathématiques*, vol. 27, *3*, pp. 527–597, 2018.

[CF18] K. Chouk and P. K. Friz, *Support theorem for a singular SPDE: The case of gPAM*, *Ann. Inst. Henri Poincaré Probab. Stat.*, vol. 54, *1*, pp. 202–219, Feb. 2018.

[CFG17] G. Cannizzaro, P. K. Friz, and P. Gassiat, *Malliavin calculus for regularity structures: The case of gPAM*, *Journal of Functional Analysis*, vol. 272, *1*, pp. 363–419, Jan. 2017.

[CH18] A. Chandra and M. Hairer, *An analytic BPHZ theorem for regularity structures*, Jan. 2018. arXiv: `1612.08138 [math-ph]`.

[CM] R. R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudo-différentiels*,

[CM23] Q. Chauleur and A. Mouzard, *The logarithmic Schrödinger equation with spatial white noise on the full space*, Aug. 2023. arXiv: `2308.15814 [math]`.

[CvZ21] K. Chouk and W. van Zuijlen, *Asymptotics of the eigenvalues of the Anderson Hamiltonian with white noise potential in two dimensions*, *Ann. Probab.*, vol. 49, *4*, pp. 1917–1964, May 2021.

[Da 06] G. Da Prato, *An Introduction to Infinite-Dimensional Analysis.* Berlin, Heidelberg: Springer, 2006.

[DD03] G. D. Da Prato and A. Debussche, *Strong solutions to the stochastic quantization equations*, *Ann. Probab.*, vol. 31, *4*, pp. 1900–1916, Oct. 2003.

[DD20] G. Da Prato and A. Debussche, *Gradient estimates and maximal dissipativity for the Kolmogorov operator in* Φ^4_2, *Electron. Commun. Probab.*, vol. 25, *none*, pp. 1–16, Jan. 2020.

[DDD19] A. Dahlqvist, J. Diehl, and B. K. Driver, *The parabolic Anderson model on Riemann surfaces*, *Probab. Theory Relat. Fields*, vol. 174, *1*, pp. 369–444, Jun. 2019.

[DDF21] A. De Bouard, A. Debussche, and R. Fukuizumi, *Two dimensional Gross-Pitaevskii equation with space-time white noise*, Jan. 2021. arXiv: `2101.10106 [math]`.

[DL20] L. Dumaz and C. Labbé, *Localization of the continuous Anderson Hamiltonian in 1-D*, *Probab. Theory Relat. Fields*, vol. 176, *1*, pp. 353–419, Feb. 2020.

[DL22] L. Dumaz and C. Labbé, *Anderson localization for the* 1*-d Schrödinger operator with white noise potential*, Dec. 2022. arXiv: `2212.04862 [cond-mat, physics:math-ph]`.

[DL24] L. Dumaz and C. Labbé, *Localization crossover for the continuous Anderson Hamiltonian in 1-d*, *Invent. math.*, vol. 235, *2*, pp. 345–440, Feb. 2024.

[DLTV23] A. Debussche, R. Liu, N. Tzvetkov, and N. Visciglia, *Global well-posedness of the 2D nonlinear Schrödinger equation with multiplicative spatial white noise on the full space*, Mar. 2023. arXiv: `2301.10825 [math]`.

[DM19] A. Debussche and J. Martin, *Solution to the stochastic Schrödinger equation on the full space*, *Nonlinearity*, vol. 32, *4*, p. 1147, Mar. 2019.

[DNY19] Y. Deng, A. R. Nahmod, and H. Yue, *Invariant Gibbs measures and global strong solutions for nonlinear Schrödinger equations in dimension two*, Oct. 2019. arXiv: `1910.08492 [math-ph]`.

[DT23] G. Da Prato and L. Tubaro, *Wick Powers in Stochastic PDEs: An Introduction*, *in Quantum Stoch. Math. Phys.*, A. Hilbert, E. Mastrogiacomo, S. Mazzucchi, B. Rüdiger, and S. Ugolini, Eds., ser. Springer Proceedings in Mathematics & Statistics, Cham: Springer International Publishing, 2023, pp. 1–15.

[DW18] A. Debussche and H. Weber, *The Schrödinger equation with spatial white noise potential*, *Electron. J. Probab.*, vol. 23, *none*, pp. 1–16, Jan. 2018.

[DZ14] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions* (Encyclopedia of Mathematics and Its Applications), 2nd ed. Cambridge: Cambridge University Press, 2014.

[DZ96] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems* (London Mathematical Society Lecture Note Series). Cambridge: Cambridge University Press, 1996.

[DZ97] G. Da Prato and J. Zabczyk, *Differentiability of the Feynman-Kac semigroup and a control application*, 1997.

[EMR24] H. Eulry, A. Mouzard, and T. Robert, *Anderson stochastic quantization equation*, Jan. 2024. arXiv: `2401.12742 [math]`.

[FG19] M. Furlan and M. Gubinelli, *Paracontrolled quasilinear SPDEs*, *Ann. Probab.*, vol. 47, *2*, pp. 1096–1135, Mar. 2019.

[FH20] P. K. Friz and M. Hairer, *A Course on Rough Paths: With an Introduction to Regularity Structures* (Universitext). Cham: Springer International Publishing, 2020.

[FN77] M. Fukushima and S. Nakao, *On spectra of the Schrödinger operator with a white Gaussian noise potential*, *Z. Wahrscheinlichkeitstheorie verw Gebiete*, vol. 37, *3*, pp. 267–274, 1977.

[GH19] M. Gubinelli and M. Hofmanová, *Global Solutions to Elliptic and Parabolic Φ^4Models in Euclidean Space*, *Commun. Math. Phys.*, vol. 368, pp. 1201–1266, Jun. 2019.

[Gho08] N. Ghoussoub, *Self-Dual Partial Differential Systems and Their Variational Principles*. Springer Science & Business Media, Nov. 2008.

[GIP15] M. Gubinelli, P. Imkeller, and N. Perkowski, *PARACONTROLLED DISTRIBUTIONS AND SINGULAR PDES*, *Forum Math. Pi*, vol. 3, e6, Aug. 2015.

[GKO18] M. Gubinelli, H. Koch, and T. Oh, *Renormalization of the two-dimensional stochastic nonlinear wave equations*, *Trans. Amer. Math. Soc.*, vol. 370, *10*, pp. 7335–7359, Jun. 2018.

[GKO23] M. Gubinelli, H. Koch, and T. Oh, *Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity*, *J. Eur. Math. Soc.*, Jan. 2023.

[GKOT22] M. Gubinelli, H. Koch, T. Oh, and L. Tolomeo, *Global Dynamics for the Two-dimensional Stochastic Nonlinear Wave Equations*, *International Mathematics Research Notices*, vol. 2022, *21*, pp. 16 954–16 999, Oct. 2022.

[GP17] M. Gubinelli and N. Perkowski, *An introduction to singular SPDEs*, Feb. 2017. arXiv: `1702.03195 [math]`.

[GUZ20] M. Gubinelli, B. Ugurcan, and I. Zachhuber, *Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions*, *Stoch PDE: Anal Comp*, vol. 8, *1*, pp. 82–149, Mar. 2020.

[Hai14] M. Hairer, *A theory of regularity structures*, *Invent. math.*, vol. 198, *2*, pp. 269–504, Nov. 2014.

[HL15] M. Hairer and C. Labbé, *A simple construction of the continuum parabolic Anderson model on* $\mathbf{R}^2$, *Electron. Commun. Probab.*, vol. 20, *none*, Jan. 2015. arXiv: `1501.00692 [math]`.

[HL17] M. Hairer and C. Labbé, *The reconstruction theorem in Besov spaces*, *Journal of Functional Analysis*, vol. 273, *8*, pp. 2578–2618, Oct. 2017.

[HL18] M. Hairer and C. Labbé, *Multiplicative stochastic heat equations on the whole space*, *J. Eur. Math. Soc.*, vol. 20, *4*, pp. 1005–1054, Mar. 2018.

[HL20] F. Hiroshima and J. Lörinczi, *Volume 2 Applications in Rigorous Quantum Field Theory*, *in Volume 2 Applications in Rigorous Quantum Field Theory*, De Gruyter, Mar. 2020.

[HL24] Y.-S. Hsu and C. Labbé, *Construction and spectrum of the Anderson Hamiltonian with white noise potential on* $\mathbf{R}^2$ *and* $\mathbf{R}^3$, Jan. 2024. arXiv: `2401.17900 [math]`.

[HM17] M. Hairer and J. Mattingly, *The strong Feller property for singular stochastic PDEs*, Apr. 2017. arXiv: `1610.03415 [math-ph]`.

[IORT23] R. Ignat, F. Otto, T. Ried, and P. Tsatsoulis, *Variational methods for a singular SPDE yielding the universality of the magnetization ripple*, *Commun. Pure Appl. Math.*, vol. 76, *11*, pp. 2959–3043, 2023.

[JP21] A. Jagannath and N. Perkowski, *A simple construction of the dynamical* Φ^4_3 *model*, Aug. 2021. arXiv: `2108.13335 [math-ph]`.

[Kön16] W. König, *The Parabolic Anderson Model: Random Walk in Random Potential* (Pathways in Mathematics). Cham: Springer International Publishing, 2016.

[KPvZ22] W. König, N. Perkowski, and W. van Zuijlen, *Longtime asymptotics of the two-dimensional parabolic Anderson model with white-noise potential*, *Ann. Inst. Henri Poincaré Probab. Stat.*, vol. 58, *3*, pp. 1351–1384, Aug. 2022.

[KPZ86] M. Kardar, G. Parisi, and Y.-C. Zhang, *Dynamic Scaling of Growing Interfaces*, *Phys. Rev. Lett.*, vol. 56, *9*, pp. 889–892, Mar. 1986.

[Lab19] C. Labbé, *The continuous Anderson hamiltonian in* $d \leq 3$, *Journal of Functional Analysis*, vol. 277, *9*, pp. 3187–3235, Nov. 2019.

[LHB20] J. Lörinczi, F. Hiroshima, and V. Betz, *Volume 1 Feynman-Kac-Type Formulae and Gibbs Measures*, *in Volume 1 Feynman-Kac-Type Formulae and Gibbs Measures*, De Gruyter, Jan. 2020.

[LOT23] P. Linares, F. Otto, and M. Tempelmayr, *The structure group for quasilinear equations via universal enveloping algebras*, *Comm. Amer. Math. Soc.*, vol. 3, *01*, pp. 1–64, 2023.

[LOTT22] P. Linares, F. Otto, M. Tempelmayr, and P. Tsatsoulis, *A diagram-free approach to the stochastic estimates in regularity structures*, Jun. 2022. arXiv: `2112.10739 [math]`.

[LRS88] J. L. Lebowitz, H. A. Rose, and E. R. Speer, *Statistical mechanics of the nonlinear Schrödinger equation*, *J Stat Phys*, vol. 50, *3*, pp. 657–687, Feb. 1988.

[Mét08] G. Métivier, *Para-Differential Calculus and Applications to the Cauchy Problem for Nonlinear Systems*. Scuola Normale Superiore, Jul. 2008.

[MM21] L. Morin and A. Mouzard, *2D random magnetic Laplacian with white noise magnetic field*, Jul. 2021. arXiv: `2101.05020 [math-ph]`.

[MO23] A. Mouzard and E. M. Ouhabaz, *A simple construction of the Anderson operator via its quadratic form in dimensions two and three*, Sep. 2023. arXiv: `2309.02821 [math]`.

[Mou21] A. Mouzard, *Weyl law for the Anderson Hamiltonian on a two-dimensional manifold*, Jul. 2021. arXiv: `2009.03549 [math-ph]`.

[Mou22] A. Mouzard, *The infinitesimal generator of the Brox diffusion*, Dec. 2022. arXiv: `2212.10840 [math]`.

[MP19] J. Martin and N. Perkowski, *Paracontrolled distributions on Bravais lattices and weak universality of the 2d parabolic Anderson model*, *Ann. Inst. Henri Poincaré Probab. Stat.*, vol. 55, *4*, pp. 2058–2110, Nov. 2019.

[MP20] J. Martin and N. Perkowski, *A Littlewood-Paley description of modelled distributions*, *Journal of Functional Analysis*, vol. 279, *6*, p. 108 634, Oct. 2020.

[Mv23] T. Matsuda and W. van Zuijlen, *Anderson Hamiltonians with singular potentials*, Jul. 2023. arXiv: `2211.01199 [math]`.

[MW17a] J.-C. Mourrat and H. Weber, *Global well-posedness of the dynamic Φ^4 model in the plane*, *Ann. Probab.*, vol. 45, *4*, pp. 2398–2476, Jul. 2017.

[MW17b] J.-C. Mourrat and H. Weber, *The Dynamic Φ^4_3 Model Comes Down from Infinity*, *Commun. Math. Phys.*, vol. 356, *3*, pp. 673–753, Dec. 2017.

[MZ22] A. Mouzard and I. Zachhuber, *Strichartz inequalities with white noise potential on compact surfaces*, Jul. 2022. arXiv: `2104.07940 [math-ph]`.

[Nua06] D. Nualart, *The Malliavin Calculus and Related Topics* (Probability, Its Applications). Berlin/Heidelberg: Springer-Verlag, 2006.

[ORSW21] T. Oh, T. Robert, P. Sosoe, and Y. Wang, *Invariant Gibbs dynamics for the dynamical sine-Gordon model*, *Proc. R. Soc. Edinb. Sect. Math.*, vol. 151, *5*, pp. 1450–1466, Oct. 2021.

[ORT23] T. Oh, T. Robert, and N. Tzvetkov, *Stochastic nonlinear wave dynamics on compact surfaces*, *Ann. Henri Lebesgue*, vol. 6, pp. 161–223, 2023.

[ORTW20] T. Oh, T. Robert, N. Tzvetkov, and Y. Wang, *Stochastic quantization of Liouville conformal field theory*, May 2020. arXiv: `2004.04194 [math-ph]`.

[ORW21] T. Oh, T. Robert, and Y. Wang, *On the Parabolic and Hyperbolic Liouville Equations*, *Commun. Math. Phys.*, vol. 387, *3*, pp. 1281–1351, Nov. 2021.

[OSSW23] F. Otto, J. Sauer, S. Smith, and H. Weber, *A priori bounds for quasilinear SPDEs in the full sub-critical regime*, Jul. 2023. arXiv: `2103.11039 [math]`.

[OT18] T. Oh and L. Thomann, *A pedestrian approach to the invariant Gibbs measures for the 2-d defocusing nonlinear Schrödinger equations*, *Stoch PDE: Anal Comp*, vol. 6, *3*, pp. 397–445, Sep. 2018.

[OT20] T. Oh and L. Thomann, *Invariant Gibbs measures for the 2-d defocusing nonlinear wave equations*, *Ann. Fac. Sci. Toulouse Mathématiques*, vol. 29, *1*, pp. 1–26, 2020.

[OTWZ22] T. Oh, L. Tolomeo, Y. Wang, and G. Zheng, *Hyperbolic $P(\Phi)_2$-model on the plane*, Nov. 2022. arXiv: `2211.03735 [math-ph]`.

[OW19] F. Otto and H. Weber, *Quasilinear SPDEs via Rough Paths*, *Arch Rational Mech Anal*, vol. 232, *2*, pp. 873–950, May 2019.

[PV23] N. Perkowski and W. Van Zuijlen, *Quantitative Heat-Kernel Estimates for Diffusions with Distributional Drift*, *Potential Anal*, vol. 59, *2*, pp. 731–752, Aug. 2023.

[PW81] G. Parisi and Y.-s. Wu, *Perturbation Theory Without Gauge Fixing*, *Sci. Sin.*, vol. 24, p. 483, 1981.

[PWZ33] R. E. A. C. Paley, N. Wiener, and A. Zygmund, *Notes on random functions*, *Math Z*, vol. 37, *1*, pp. 647–668, Dec. 1933.

[PZ30] R. E. a. C. Paley and A. Zygmund, *On some series of functions, (1)*, *Math. Proc. Camb. Philos. Soc.*, vol. 26, *3*, pp. 337–357, Jul. 1930.

[PZ32] R. E. a. C. Paley and A. Zygmund, *On some series of functions, (3)*, *Math. Proc. Camb. Philos. Soc.*, vol. 28, *2*, pp. 190–205, Apr. 1932.

[RS03] M. Reed and B. Simon, *III: Scattering Theory*, Nachdr. New York: Acad. Pr, 2003.

[RS75] M. Reed and B. Simon, *II: Fourier Analysis, Self-Adjointness*. Elsevier, 1975.

[RS78] M. Reed and B. Simon, *IV: Analysis of Operators*. Elsevier Science, Apr. 1978.

[RS81] M. Reed and B. Simon, *I: Functional Analysis*. Academic Press, Feb. 1981.

[RY04] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*. Springer Science & Business Media, Sep. 2004.

[RZ] T. Robert and Y. Zine, *Stochastic complex Ginzburg-Landau equation on compact surfaces*, *preprint*,

[She21] H. Shen, *Stochastic Quantization of an Abelian Gauge Theory*, *Commun. Math. Phys.*, vol. 384, *3*, pp. 1445–1512, Jun. 2021.

[Sim74] B. Simon, *The $P(\Phi)_2$ Euclidean (Quantum) Field Theory.* Princeton University Press, 1974.

[Tam01] J. Tambača, *Estimates of the Sobolev Norm of a Product of Two Functions*, *Journal of Mathematical Analysis and Applications*, vol. 255, *1*, pp. 137–146, Mar. 2001.

[Tre19] W. J. Trenberth, *Global well-posedness for the two-dimensional stochastic complex Ginzburg-Landau equation*, Nov. 2019. arXiv: `1911.09246 [math]`.

[TV23a] N. Tzvetkov and N. Visciglia, *Global dynamics of the 2d NLS with white noise potential and generic polynomial nonlinearity*, *Commun. Math. Phys.*, vol. 401, *3*, pp. 3109–3121, Aug. 2023. arXiv: `2204.03280 [math]`.

[TV23b] N. Tzvetkov and N. Visciglia, *Two dimensional nonlinear Schrödinger equation with spatial white noise potential and fourth order nonlinearity*, *Stoch PDE: Anal Comp*, vol. 11, *3*, pp. 948–987, Sep. 2023.

[TW18] P. Tsatsoulis and H. Weber, *Spectral gap for the stochastic quantization equation on the 2-dimensional torus*, *Ann. Inst. Henri Poincaré Probab. Stat.*, vol. 54, *3*, pp. 1204–1249, Aug. 2018.

[Tzv08] N. Tzvetkov, *Invariant measures for the defocusing Nonlinear Schrödinger equation*, *Ann. Inst. Fourier*, vol. 58, *7*, pp. 2543–2604, 2008.

[Uek] N. Ueki, *A definition of self-adjoint operators derived from the Schrödinger operator with the white noise potential on the plane*, *preprint*,

[Ugu22] B. E. Ugurcan, *Anderson Hamiltonian and associated Nonlinear Stochastic Wave and Schrödinger equations in the full space*, Aug. 2022. arXiv: `2208.09352 [math]`.

[Üst14] A. S. Üstünel, *Variational calculation of Laplace transforms via entropy on Wiener space and applications*, *Journal of Functional Analysis*, vol. 267, *8*, pp. 3058–3083, Oct. 2014.

[ÜZ00] A. S. Üstünel and M. Zakai, *Transformation of Measure on Wiener Space* (Springer Monographs in Mathematics). Berlin, Heidelberg: Springer, 2000.

[Wil96] M. Willem, *Minimax Theorems.* Boston, MA: Birkhäuser, 1996.

[Zac20] I. Zachhuber, *Strichartz estimates and low-regularity solutions to multiplicative stochastic NLS*, Oct. 2020. arXiv: `1911.01982 [math]`.

[ZD23] Q. Zhang and J. Duan, *Constraint Minimization Problem of the Nonlinear Schrödinger Equation with the Anderson Hamiltonian*, May 2023. arXiv: `2111.10313 [math]`.

COLLEGE DOCTORAL BRETAGNE — MATHS, TELECOMS, INFORMATIQUE, SIGNAL, SYSTEMES, ELECTRONIQUE

Université de Rennes

Titre : Quantification stochastique d'Anderson et calcul paracontrôlé

Mots clés : EDPS singulières, calcul paracontrôlé, opérateur d'Anderson, mesures de Gibbs

Résumé : Cette thèse porte sur l'étude d'équations aux dérivées partielles dirigées par un opérateur aléatoire singulier. L'objet central de ce travail est l'opérateur d'Anderson, c'est-à-dire l'opérateur de Schrödinger avec comme potentiel un bruit blanc spatial. En utilisant les outils du calcul paracontrôlé et après une procédure de renormalisation, on est à même de définir cet opérateur singulier sur une surface compacte et d'obtenir de bonnes propriétés spectrales, en particulier une compréhension fine de sa fonction de Green est proposée. On peut alors par exemple adapter des méthodes de compacité et de fonctionnelles auto-duales pour étudier l'existence de solutions aux équations stationnaires dirigées par cet opérateur sur une surface compacte. On s'intéresse également à une version non-locale et quasilinéaire du modèle parabolique d'Anderson sur le tore, en établissant un résultat d'existence locale sous certaines conditions.

Une large partie de ce manuscrit est dédiée à la question de la quantification stochastique dans l'environnement singulier régi par l'opérateur d'Anderson. À l'aide de la formule de Boué-Dupuis, on construit une mesure de Gibbs pour l'équation de quantification associée à cet opérateur et on étudie le caractère bien posé de l'équation. On s'intéresse ensuite à une construction dynamique de cette mesure dans le cas de l'équation Φ^4_2 dirigée par l'opérateur d'Anderson, pour obtenir de bonnes propriétés probabilistes sur la solution.

Title: Anderson Stochastic Quantization and paracontrolled calculus

Keywords: Singular SPDEs, paracontrolled calculus, Anderson Hamiltonian, Gibbs measures

Abstract: This thesis focuses on studying partial differential equations driven by a singular random operator. The central object of this work is the Anderson operator, namely the Schrödinger operator with spatial white noise as its potential. By employing tools from paracontrolled calculus and through a renormalization procedure, we are able to define this singular operator on a closed surface and obtain good spectral properties. In particular, a detailed understanding of its Green's function is provided. We can then adapt compactness and convex analysis methods to study the existence of solutions to stationary equations driven by this operator on a closed surface. Additionally, we investigate a non-local and quasilinear version of the parabolic Anderson model on the torus, establishing a result of local existence under certain conditions.

A substantial part of this manuscript is devoted to the question of the stochastic quantification in the singular environment governed by the Anderson operator. Using the Boué-Dupuis formula, we construct a Gibbs measure for the quantification equation associated with this operator and investigate the well-posedness of the equation. We then focus on a dynamic construction of this measure in the case of the Φ^4_2 equation driven by the Anderson operator, aiming to obtain nice probabilistic properties of the solution.